Space of marked groups and non-uniform exponential growth

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Let $\mathfrak{G}_n$ be the set of marked n-generated groups. It can be identified with the set of normal subgroups of F_n.

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The induced topology on $\mathfrak{G}_n$ is given by the basis of open sets:

$$\mathcal{U}_{A,B} = \{N \lhd F_n \ : \ A \subset N, \quad B \cap N = \emptyset\},$$

where A and B are finite subsets of F_n.

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The function $e : \mathfrak{G}_n \longrightarrow \mathbb{R} : (G, g_1, \ldots, g_n) \mapsto e_{\{g_1,\ldots,g_n\}}(G)$ is upper semi-continuous.

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Proposition

Let $G_k = (G, g_{1,k}, \ldots g_{n,k}) \in \mathfrak{G}_n$ *be a sequence of isomorphic groups of exponential growth such that* $H = \lim_{k\to\infty} G_k$ *has sub-exponential growth.*

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Then G has non-uniform exponential growth and H has intermediate growth.

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$$\begin{aligned} f_i(\mathrm{A}_i) &= \mathrm{B}_{i-1} \\ f_i(\mathrm{B}_i) &= \Gamma_{i-1} \\ f_i(\Gamma_i) &= \mathrm{B}_{i-1}. \end{aligned}$$

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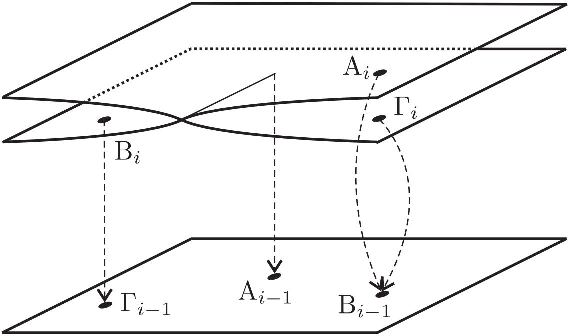
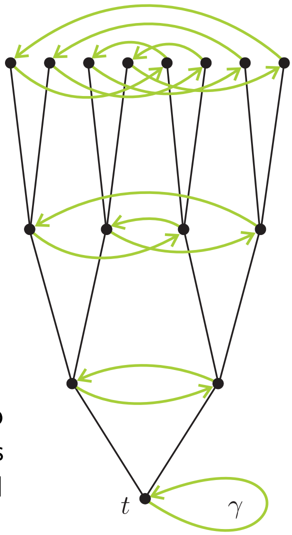


The fundamental group of $C_0 \setminus \{\mathrm{A}_0, \mathrm{B}_0, \Gamma_0\}$ acts on the tree of backward orbit of the basepoint.



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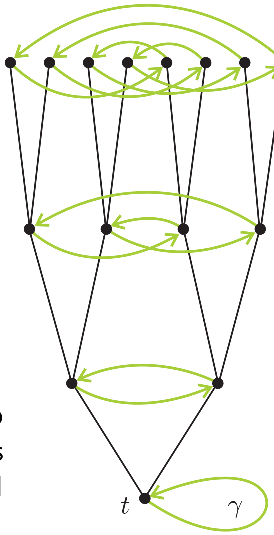

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The quotient of the fundamental group by the kernel of the action is the *iterated monodromy group* of the iteration.

Let us fix the bottom plane C_0 together with a generating set $\{\alpha, \beta, \gamma\}$ of $\pi_1(C_0 \setminus \{\mathrm{A}_0, \mathrm{B}_0, \Gamma_0\}, t)$ and consider all possible backward iterations of the described form.

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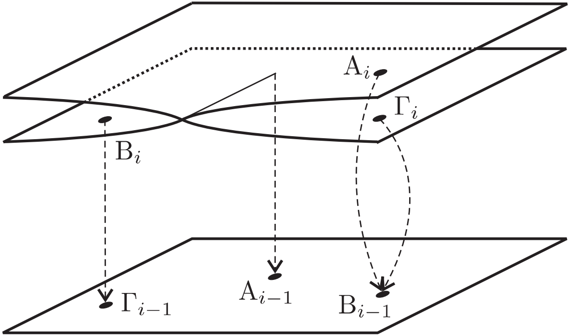
The mapping class group of the punctured plane acts on this set by post-compositions with f_0, leaving the isomorphism classes invariant.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$$\begin{aligned}
\alpha_w(xv) &= \overline{x}v, \\
\beta_w(0v) &= 0\alpha_{s(w)}(v), \\
\beta_w(1v) &= 1\gamma_{s(w)}(v) \\
\gamma_w(x_1 v) &= x_1\beta_{s(w)}(v), \\
\gamma_w(\overline{x_1} v) &= \overline{x_1} v,
\end{aligned}$$

Here $w = x_1 x_2 x_3 \ldots$ and $s(w) = x_2 x_3 x_4 \ldots$.

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4. w_1 *and* w_2 *are cofinal.*

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$$\exp\left(\frac{n}{\log^{2+\epsilon}(n)}\right) \preceq \Gamma(n) \preceq \exp\left(\frac{n}{\log^{1-\epsilon}(n)}\right).$$

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The map $\Omega \longrightarrow \mathfrak{G}_3 : w \mapsto (G_w, \alpha_w, \beta_w, \gamma_w)$ *is a homeomorphic embedding.*

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The generators of G_w for $w \in \Omega$ satisfy the relations

$$\alpha_w^2 = \beta_w^2 = \gamma_w^2 = (\alpha_w \beta_w)^8 = (\alpha_w \gamma_w)^4 = (\beta_w \gamma_w)^8 = 1,$$

while in $G_{111\ldots}$ we have

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In particular, the isomorphism classes are dense in $\widetilde{\mathcal{G}}$.

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The set of growth types of groups G_w *is uncountable and contains uncountable chains and anti-chains.*

Open problem

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$$\begin{aligned} \alpha &= \sigma(1, \gamma), \\ \beta &= (1, \alpha), \\ \gamma &= (\beta, 1). \end{aligned}$$

It is locally isomorphic to the “rabbit” group

$$\begin{aligned} \alpha &= \sigma(1, \gamma), \\ \beta &= (1, \alpha), \\ \gamma &= (1, \beta), \end{aligned}$$

which contains a free subsemigroup.