Cohomology of Coxeter groups and buildings

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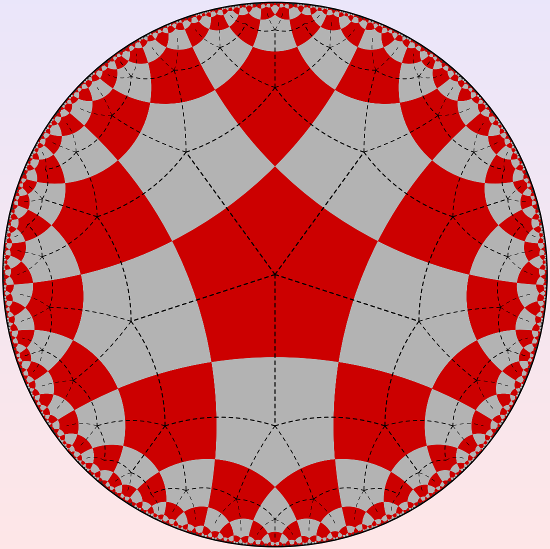


The theory of abstract reflection groups or Coxeter groups was developed by J. Tits around 1960. This is a much larger class of groups than the classical examples of geometric reflection gps on spaces of constant curvature. Nevertheless, associated to any Coxeter gp W there is a contractible simplicial cx Σ on which W acts properly and cocompactly.

Tits coined the term “Coxeter gp” in connection with the theory of buildings. Actually, the bldgs of interest to algebraists all are associated either to finite reflection gps on spheres (leading to spherical bldgs) or to Euclidean reflection gps (leading to affine bldgs).

In fact, ∃ bldgs associated to general Coxeter gps and each has an associated contractible cx Λ, similar to Σ. These give interesting examples for geometric group theory.

We will give a formula for the compactly supported cohomology of Σ and Λ. (In some cases this formula is not yet proved.) This means we consider cellular cochains on these complexes with finite support. One can impose other conditions on cellular cochains, e.g., that they be square summable. This leads to L^2-cohomology.

Chamber systems

Definition

A *chamber system* is a set Φ (of "chambers") together with a family of equivalence relations indexed by some set S

Two chambers c and d are *s-adjacent* if they are s-equivalent and not equal.

A *gallery* in Φ is a finite sequence $(c_0, \dots, c_n)$ of adjacent chambers. It is *type* $(s_1, \dots, s_n)$ if c_i and c_{i-1} are s_i-adjacent. (So, its type is a word in S.)

Definitions

- Φ is *connected* if any two chambers can be connected by a gallery.
- Given a subset $T \subset S$, a gallery of type $(s_1, \ldots, s_n)$ is a *T-gallery* if each $s_i \in T$.
- A *residue* of *type* T is a T-gallery connected component of Φ.

Coxeter groups

A *Coxeter matrix* on a set S is an $S \times S$ symmetric matrix (m_{st}) with 1's on the diagonal and off-diagonal entries integers ≥ 2 or the symbol ∞. The associated *Coxeter group* is the group W with generating set S and relations:

$$(st)^{m_{st}} = 1$$

(Note: these relations imply that each $s \in S$ is an involution.)

The pair (W, S) is a *Coxeter system*.

For each $T \subset S$, $W_T := \langle T \rangle$. Also, W_s is the subgroup $\langle s \rangle$ of order 2 generated by s.

Example

W has the structure of a chamber system over S.

- $v, w \in W$ are s-equivalent $\iff$ they belong to the same coset of W_s.
- A gallery in W is an edge path in $\mathrm{Cay}(W, S)$. (Here a "chamber" is a vertex of $\mathrm{Cay}(W, S)$.)
- Its type is a word in S.
- A T-residue is a coset of W_T in W.

T is a *spherical* subset of S if $|W_T| < \infty$.
$\mathcal{S} :=$ the poset of spherical subsets.

Reduced expressions

Given a word $\mathbf{s} = (s_1, \dots, s_n)$ in S, its *value* is $w(\mathbf{s})$ is the element $w \in W$ defined by $w := s_1 \cdots s_n$.
$\mathbf{s}$ is a *reduced expression* for w if $l(w) = n$.

Buildings

Suppose (W, S) is a Coxeter system.

Definition

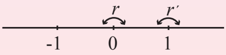
A chamber system Φ over S is a *building* of type (W, S) if each s-equivalence class has at least two elements and if it admits a *W-valued distance function* $\delta : \Phi \times \Phi \to W$.
This means that if $\mathbf{s} = (s_1, \ldots, s_n)$ is a reduced expression for w and if $c_0, c_n \in \Phi$, then there is a gallery of type $\mathbf{s}$ from c_0 to c_n $\iff \delta(c_0, c_n) = w$

Example (The edge set of a tree)

Suppose W is the infinite dihedral group with generators s, t. Let $\mathcal{T}$ be a tree w/o terminal vertices. Any tree is bipartite. Call the colors s, t.

Let $\Phi = \text{Edge}(\mathcal{T})$. An edge path in $\mathcal{T}$ gives a gallery. The sequence of colors of the vertices it crosses gives a word in $\{s, t\}$. The word is a reduced expression if the path has no backtracking.

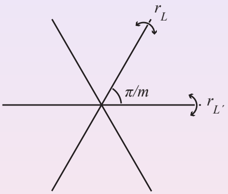
The W-distance between b,$c \in \Phi$ is defined as follows. Take the edge path w/o backtracking from b to c. Its type gives a word in $\{s, t\}$ and hence, an element $w \in W$ and $\delta(b, c) = w$.



Definitions

If W is

- finite, Φ is a *spherical bldg*.
- a cocompact Euclidean reflection gp, Φ is a *Euclidean* (or *affine*) *bldg*.
- a finite dihedral gp, Φ is a *generalized polygon*.



Example

W itself is a bldg. W-distance $\delta : W \times W \to W$ is defined by $\delta(v, w) = v^{-1}w$.

An *apartment* is the image of a W-isometric embedding $W \hookrightarrow \Phi$.

Geometric realizations

Suppose $\mathcal{P}$ is a poset. Its *geometric realization* $|\mathcal{P}|$ is the simplicial cx with one simplex for each chain $p_0 < \cdots < p_k$ in $\mathcal{P}$.

Recall T is a *spherical* subset of S if $|W_T| < \infty$ and
$\mathcal{S} :=$ poset of spherical subsets.
$W\mathcal{S} :=$ poset of spherical cosets $= \coprod W/W_T$. For a bldg Φ,
$\mathcal{C}$:= poset of spherical residues in Φ.

$$K := |\mathcal{S}| \qquad \Sigma := |W\mathcal{S}| \qquad \Lambda := |\mathcal{C}|.$$

Basic Fact

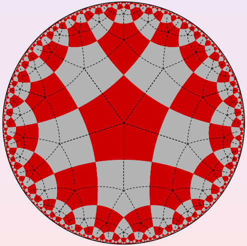
Λ (and in particular, Σ) is $\mathrm{CAT}(0)$ and hence, is contractible.

$$K := |\mathcal{S}| \qquad \Sigma := |W\mathcal{S}| \qquad \Lambda := |\mathcal{C}|.$$

W is gp generated by reflections in the faces of right-angled pentagon.

K is a pentagon.

Σ is the tessellation of $\mathbb{H}^2$ by pentagons



Another approach to defining the geometric realization of a chamber cx or bldg is to declare each chamber to be homeomorphic to a certain space X and then declare how adjacent chambers are glued together.

Usually we take $X = K$ (the geometric realization of $\mathcal{S}$). Another possibility is $X = \Delta$ a simplex with its $\{$codim 1 faces$\}$ indexed by S. In that case we get the *Coxeter complex*.

Here are the details:

Alternate description

- X a space, $\{X_s\}_{s \in S}$ a family of closed subspaces. $S(x) := \{s \in S \mid x \in X_s\}$.
- $\mathcal{U}(\Phi, X) := (\Phi \times X)/\sim$, where $(c, x) \sim (c', x') \iff x = x'$ and c,c' belong to the same $S(x)$-residue.
- $\mathcal{U}(\Phi, X)$ is formed by pasting together copies of X, one for each chamber in Φ.
- Put $K_s := |\mathcal{S}_{\geq \{s\}}|$. Then $\Lambda = \mathcal{U}(\Phi, K)$.

When $\Phi = W$, two chambers ($=$ copies of X) meet along each face of type X_s. For a general bldg there are two or more along each X_s. (Think: line versus tree.)

Right-angled bldgs (= RABs)

A Coxeter gp is *right angled* if each m_{st} is $= 2$ or ∞.
A bldg is *right-angled* if its Coxeter gp is.

Example (Product of trees)

If $W = (D_\infty)^n$, then K is an n-cube. A bldg of type (W, S) is a product of n trees.

Will show

$\forall$ right-angled (W, S), $\exists$ RABs of type (W, S) with chamber transitive automorphism gp and arbitrary thickness along faces of type s.

Graph products of type (W, S)

- (W, S) a R-A Coxeter system.
- Let L^1 be a graph with $\mathrm{Vert}(L^1) = S$ and $\mathrm{Edge}(L^1) = \{\{s, t\} \mid m_{st} \neq \infty\}$.
- Let $\{G_s\}_{s \in S}$ be a family of finite gps. For each $T \in \mathcal{S}$ (i.e., for each complete subgraph of L^1), put

$$G_T := \prod_{s \in T} G_s.$$

- $G = \varinjlim G_T$ is the *graph product* of the G_s.

Put $\Phi = G$. If $g = g_{s_1} \cdots g_{s_n} \in G$, put $\delta(1, g) = s_1 \cdots s_n \in W$ and $\delta(h, g) = \delta(1, h^{-1}g)$. Then Φ is a RAB. The number of chambers along a face of type s is $\mathrm{Card}(G_s)$.

Conclusion

RABs of arbitrary thickness exist.

Kac-Moody gps and bldgs

- Suppose each $m_{st} \in \{1, 2, 3, 4, 6, \infty\}$ and $\mathbb{F}_q$ is a finite field.
- $\exists$ a "Kac-Moody gp" of type (W, S) over $\mathbb{F}_q$. It has structure of a "(B, N) pair" and hence, gives a bldg Φ with chamber transitive action of G.
- $q + 1$ chambers meet along each face of type s.
- G acts properly on $\Lambda \times \Lambda$ (where Λ is the geometric realization of Φ.)

Compactly supported cohomology of Σ and Λ

Goal

Compute the compactly supported (i.e., finitely supported) cellular cohomology of Σ and Λ.

Why?

- $H^*_c(\Sigma) = H^*(W;\mathbb{Z}W)$. More generally, if Γ acts properly and cocompactly on Λ, then $H^*_c(\Lambda) = H^*(\Gamma;\mathbb{Z}\Gamma)$. Why do we want to know this?
- $\mathrm{rk}(H^1_c(\Gamma;\mathbb{Z}\Gamma))$ tells the number of ends of Γ.
- $\max\{n \mid H^n_c(\Gamma;\mathbb{Z}\Gamma) \neq 0\} = \mathrm{vcd}\,\Gamma$.

There is a conjectured formula for $H_c^*(\Lambda)$, not yet proved in all cases (however, it is proved for $\Phi = W$ and for Φ a RAB).

- $A := \{\text{finitely supported functions } f : \Phi \to \mathbb{Z}\}$.
- $A^T := \{f \in A \mid f \text{ is constant on all } T\text{-residues}\}$
  Note: $A^T = 0$ whenever T is not spherical.

$$A^{>T} := \sum_{U \supsetneq T} A^U \subset A^T.$$

- $D^T := A^T / A^{>T}$.
- *Remark*: When $\Phi = W$, $A^T = \{\text{functions on } W/W_T\}$, i.e., it is the set of W_T-invariant elements in $\mathbb{Z}W$.

Conjecture

D^T is free abelian. Let $\widehat{A}^T$ be image of a (nonequivariant) splitting $D^T \hookrightarrow A^T$.

$$A^T = \bigoplus_{U \supset T} \widehat{A}^U \qquad \text{and} \qquad A = \bigoplus_{U} \widehat{A}^U.$$

This Decomposition Conjecture implies a computation of $H_c^*(\ )$ (because it gives a direct sum decomposition of coefficient systems).

Given a finite CW complex X and a family of subcomplexes $\{X_s\}_{s\in S}$, recall $\mathcal{U}(\Phi, X) := (\Phi \times X)/\sim$. For any $U \subset S$, put

$$X^U := \bigcup_{s\in U} X_s.$$

Assume $\bigcap_{s\in T} X_s = \emptyset$ whenever W_T is not spherical.

Conjecture

$$H_c^*(\mathcal{U}(\Phi, X)) = \bigoplus_{T\in\mathcal{S}} H^*(X, X^{S-T}) \otimes \widehat{A}^T.$$

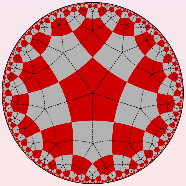
True for $\Phi = W$ or for Φ a RAB.

Corollary

If Γ acts properly and cocompactly on a bldg, then $\operatorname{vcd} \Gamma = \operatorname{vcd} W$.

Corollary

If Σ is a (contractible) mfld, then $H_c^*(\Lambda)$ is concentrated in the top degree.



L^2-cohomology

- Y a CW complex
- $L^2C^i(Y) =$ the cellular i-cochains on Y which are square summable.
- $L^2H^i(Y) := \operatorname{Ker}\delta/\operatorname{Im}\delta, \quad L^2\mathcal{H}(Y) := \operatorname{Ker}\delta/\overline{\operatorname{Im}\delta}$.
- They have "von Neumann dimensions" giving the "L^2-Betti numbers."

Growth series of W

$$W(t) := \sum_{w \in W} t^{l(w)}$$

It is the power series of an explicit rational function of t.
$\rho :=$ its radius of convergence.

Φ a bldg with $q+1$ chambers in each s-equivalence class.

Goal

Calculate L^2-Betti numbers of $\mathcal{U} = \mathcal{U}(\Phi, X)$.
We can do this for $q > \rho^{-1}$. The method is to calculate the "weighted L^2-cohomology" of Σ (the cx for W).

The L^2-Euler characteristic

The alternating sum of L^2-Betti numbers :=

$$L^2\chi(\Lambda) = \frac{1}{W(q)}.$$

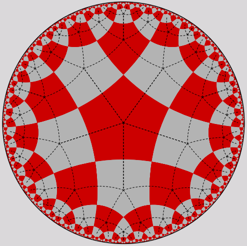
Example (W is right angled pentagon group)

$$\frac{1}{W(q)} = \frac{q^2 - 3q + 1}{(1+q)^2}$$

which has roots

$$\rho^{\pm 1} = \frac{3 \mp \sqrt{5}}{2}.$$

So, $2 < \rho^{-1} < 3$



As before

$A := \{$square summable functions on $\Phi\}$
$A^T := \{f \in A \mid f$ is constant on each T-residue$\}$
$D^T :=$ orthogonal complement of $A^{>T}$ in A^T.

Theorem (DDJO)

For $q > \rho^{-1}$,

$$L^2\mathcal{H}^*(\mathcal{U}(\Phi, X)) \cong \bigoplus_{T \in \mathcal{S}} H^*(X, X^{S-T}) \otimes D^T.$$

This is a consequence of the following

Decomposition Theorem

For $q > \rho^{-1}$,

$$A = \overline{\bigoplus_{T \in \mathcal{S}} D^T}$$

Conclusion

For $q > \rho^{-1}$, $L^2\mathcal{H}^*(\ )$ looks like $H_c^*(\ )$.

Example

If Σ is a 2-mfld, then $L^2\mathcal{H}^*(\Lambda)$ is concentrated in degree:

$$\begin{cases} 1, & \text{if } q < \rho^{-1}; \\ 2, & \text{if } q > \rho^{-1}. \end{cases}$$

When K is a right-angled pentagon, $\rho^{-1} = \frac{3+\sqrt{5}}{2}$.