# Teichmüller Space and (one of) its metrics

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• Teichmüller Space and the Teichmüller metric

Let $M$ be a closed oriented surface of genus $g \geq 2$. The Teichmüller space of $M$ is defined as follows:

$$\mathcal{T}(M) := \{\text{complex structures } X \text{ on } M\}/\sim$$

where $X_1 \sim X_2 \iff$ there exists a biholomorphic map $h : X_1 \to X_2$ which is isotopic to the identity as a topological map $M \to M$.

The Teichmüller metric is given by $d(X_1, X_2) = \log K$, where $K$ is the smallest number for which there exists a $K$-quasiconformal map $X_1 \to X_2$ which is isotopic to the identity on $M$.

Roughly speaking, a map is $K$-quasiconformal if locally it maps circles to ellipses where the ratio of the major axis to the minor axis is bounded above by $K$.

Teichmüller's theorem guarantees that for any $X_1, X_2 \in \mathcal{T}$, a smallest $K$ exists.

As an example, we can consider the Teichmüller space of the once punctured torus. In this case, $\mathcal{T}(M)$ can be identified with the upper half plane, and the Teichmüller metric coincides with the Poincaré metric.

• Isometries of Teichmüller space

The group of isometries of the Teichmüller metric may be identified with with group of orientation preserving diffeomorphisms of $M$ modulo diffeomorphisms isotopic to the identity:

$$\text{Isom}(\mathcal{T}(M)) = \text{Diff}^+(M)/\text{Diff}_0(M)$$

By work of Royden, Ivanov and others, this group may be identified with the mapping class group of $M$.

• Teichmüller geodesics

In order to describe geodesics in the Teichmüller metric we need to mention quadratic differentials. A holomorphic quadratic differential $q$ on $X \in \mathcal{T}$ is an assignment of a holomorphic function $q(z)$ to each chart $(U, z)$, such that if $(U, z)$ and $(U', z')$ are overlapping charts

$$q_z(z)dz^2 = q_{z'}(z')dz'^2$$

A quadratic differential defines a flat metric on $X$ which is given locally by $|q(z)|^{\frac{1}{2}}|dz|$ (with singularities at the zeroes of $q$). $q$ also defines a pair of transverse foliations as follows. We may define the horizontal trajectories of $q$ to be arcs on which $q(z)dz^2 > 0$, and the vertical trajectories of $q$ to be arcs on which $q(z)dz^2 < 0$. The vertical foliation has leaves that are vertical trajectories, and the horizontal foliation has leaves that are horizontal trajectories.

Away from the zeroes of $q(z)$ we can define a parameter $\omega = x+iy$ such that $d\omega^2 = q(z)dz^2$). The flat metric is then the pullback of the Euclidean metric on $\mathbb{C}$.

We can now describe a Teichmüller geodesic. Given $X \in \mathcal{T}(M)$, $q$ a quadratic differential and $z = x + iy$ the natural parameter for $q$, define $z_t = e^{-\frac{t}{2}}x + ie^{\frac{t}{2}}y$. $z_t$ defines a new complex structure for the Riemann surface, and hence gives a point $X_t \in \mathcal{T}(m)$ together with a quadratic differential $q_t$.

Teichmüller's theorem can be used to show that the trajectory so defined is a geodesic in $\mathcal{T}(M)$ with respect to the Teichmüller metric.

• Compactifications of Teichmüller space

$\mathcal{T}(M)$ with the Teichmüller metric is a complete geodesic metric space that is noncompact. We will discuss two compactifications of this space, the Teichmüller compactification and Thurston's compactification by the space of projective measured foliations.

The Teichmüller compactification is defined as follows. Fix a basepoint $X_0 \in \mathcal{T}(M)$, and consider all geodesic rays from $X_0$. Considering these rays as points in the visual boundary gives the Teichmüller compactification of the $\mathcal{T}(M)$. This compactification lacks desired naturality properties, as the compactification given depends heavily on the basepoint $X_0$. Kerckhoff showed that the action of the mapping class group does not extend continuously to the Teichmüller boundary.

Thurston found another natural compactification of Teichmüller space for which the action of the mapping class group does extend to the boundary. This is the compactification by the space of projective measured foliations, $\mathcal{PMF}(M)$.

Let $S$ be the set of isotopy classes of essential simple closed curves, and let $R_+^S$ be the space of positive functions on $S$ with the product topology, $\mathbb{P}R_+^S$ the corresponding projective space.

Via the uniformization theorem, $\mathcal{T}(M)$ may also be considered as the space of hyperbolic structures $(M, \rho)$ on $M$ modulo an equivalence relation $\sim$, where $(M, \rho_1) \sim (M, \rho_2)$ if there exists an isometry from $(M, \rho_1) \to (M, \rho_2)$ that is isotopic to the identity. Using this interpretation of Teichmüller space, we get a map $\mathcal{T}(M) \to \mathbb{P}R_+^S$ sending $(M, \rho) \mapsto (F_\rho)$, where $F_\rho(\gamma)$ is the length of $\gamma$ on $(M, \rho)$.

The image of $\mathcal{T}(M)$ in $\mathbb{P}R_+^S$ is homeomorphic to an open ball in $\mathbb{R}^{6g-6}$ where $g$ is the genus of $M$, and the boundary corresponds to $\mathcal{PMF}(M)$, a $6g-7$ dimensional sphere. $\mathcal{T}(M) \cup \mathcal{PMF}(M)$ is the Thurston compactification of Teichmüller space.

• Questions and answers about the Teichmüller metric

**1.** Is $\mathcal{T}(M)$ with the Teichmüller metric non-positively curved in the sense of Busemann, i.e. for any geodesic triangle PQR do the midpoints A of PQ and B of PR satisfy d(A,B)$\leq \frac{1}{2}$d(Q,R)?

Masur answered this question in the negative by discovering geodesic rays from a single point that stay a bounded distance apart.

**2.** Is the space Gromov hyperbolic?

No. (Masur, Wolf)

**3.** In hyperbolic space, the rate of divergence of two diverging rays is exponential, is this the case in $\mathcal{T}(M)$?

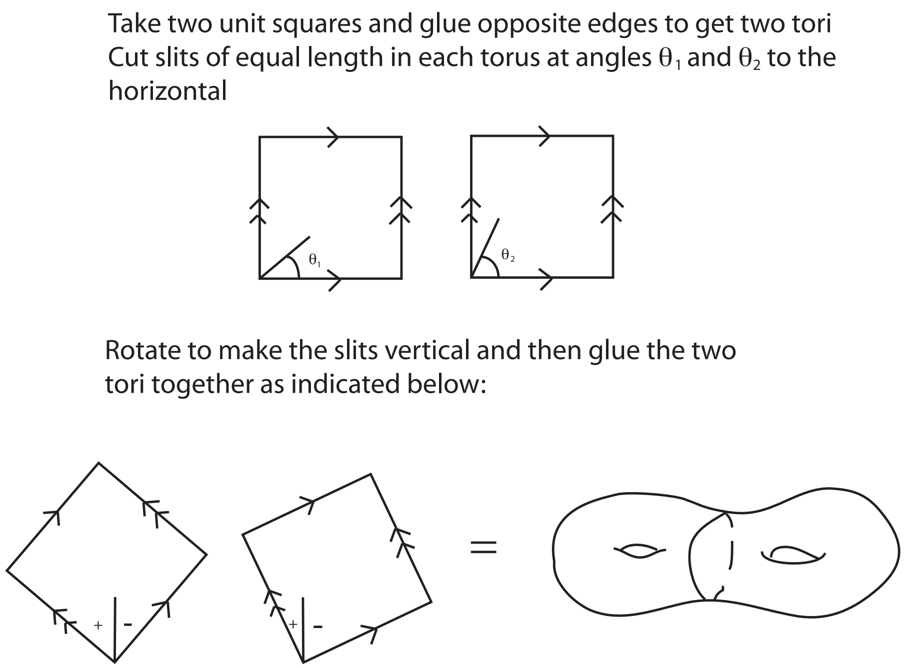
No. In the Teichmüller metric this divergence is at most quadratic (Rafi, Duchin).

**4.** What is the asymptotic behavior of Teichmüller geodesics with respect to Thurston's compactification? Given a ray $(X, q)$, does it have a limit in $\mathcal{PMF}(M)$?

(i) The answer to this question depends on the vertical foliation $F_v$. If $F_v$ is uniquely ergodic, then Masur has shown that $X_t$ converges to $[F_v]$.

(ii) If $F_v$ is such that all leaves are closed, Masure has shown that $X_t$ converges to $[F]$, where $F$ is topologically the same as $F_v$ but all leaves are given equal weights.

(iii) Not all geodesic rays have a limit. The following diagram shows a construction that gives a genus two surface depending on two parameters $\theta_1$ and $\theta_2$:




We can contract along the vertical foliation by $e^{-\frac{t}{2}}$ and stretch along the horizontal foliation by $e^{\frac{t}{2}}$ to get a geodesic ray $(X_t)_{t>0}$. If $\theta_1$ and $\theta_2$ are both irrational, and the coefficients in the continued fraction decomposition of $\theta_1$ are bounded while those of $\theta_2$ are unbounded, then the ray does not converge in $\mathcal{PMF}(M)$.