Fixed points for centralizers and Morita's theorem

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The Lifting Problem

The *mapping class group* MCG(S) of a surface S is the group of isotopy classes of orientation preserving homeomorphisms of S.

There is a natural homomorphism

$$\text{Homeo}(S) \to \text{MCG}(S)$$

A *lift* of a subgroup Γ of MCG(S) is a homomorphism

$$\Gamma \to \text{Homeo}(S)$$

so that the composition

$$\Gamma \to \text{Homeo}(S) \to \text{MCG}(S)$$

is the inclusion.

Question 1. *Which subgroups* Γ *of* MCG(S) *lift to* Homeo(S)[Diff(S)]*?*

If genus(S) = 1, MCG(S) lifts to Diff(S) so assume that genus(S) $\geq$ 2.

- Any free group

- Any free abelian group

- Any finite group [Kerckhoff]

- MCG(S) does not lift to Diff(S) for genus(S) $\geq$ 5 [Morita]

- MCG(S) does not lift to Homeo(S) for genus(S) $\geq$ 6 [Markovic]

Question 2. *Which subgroups of* MCG(S) *act faithfully on* S *by diffeomorphisms [homeomorphisms]?*

Question 3. *Is there a finite index subgroup of* MCG(S) *that acts faithfully by diffeomorphisms [homeomorphisms] that are isotopic to the identity?*

We have a new elementary proof of Morita's Theorem and some partial results about other actions.

Theorem 4. MCG(S) *does not lift to* Diff(S) *for* genus(S) ≥ 3.

Notation and General Statement

Divide S along a simple closed curve into subsurfaces M_1 with genus(M_1) ≥ 2 and M_2 with genus(M_2) ≥ 1.

MCG($M_i, \partial M_i$) is the set of isotopy classes rel ∂M_i of homeomorphisms of M_i that are the identity on ∂M_i.

MCG($M_i, \partial M_i$) $\hookrightarrow$ MCG(S).

MCG($M_1, \partial M_1$) commutes with MCG($M_2, \partial M_2$)

We denote the isotopy class of a homeomorphism h by $[h]$.

Theorem 5. *Suppose that* genus$(S) \geq 3$ *and that* $\Gamma = \langle \Gamma_1, \alpha \rangle$ *where*

- Γ_1 *is a non-trivial subgroup of* $\mathsf{MCG}(M_1, \partial M_1)$ *such that* $H^1(\Gamma_1)$ *has rank zero.*
- $\alpha \in \mathsf{MCG}(M_2, \partial M_2)$*.*

Then there does not exist a homomorphism $\mathcal{L} : \Gamma \to \mathsf{Diff}(S)$ *such that*

1. $[\mathcal{L}(\Gamma_1)] \subset \mathsf{MCG}(M_1, \partial M_1))$*.*
2. $[\mathcal{L}(\alpha)] \in \mathsf{MCG}(M_2, \partial M)$ *is represented by* $A : S \to S$ *where* $A|M_2$ *is pseudo-Anosov.*

$H^1(\mathrm{MCG}(M_1, \partial M_1))$ has rank zero ([Korkmaz])

Morita also proved that finite index subgroups of MCG(S) do not lift.

Question 6. *Does $H^1(\Gamma)$ have rank zero for every finite index subgroup of* MCG(S), (genus(S) $\geq$ 3)*?*

Question 7. *Let M be the genus one surface with one boundary component. Does* MCG($M, \partial M$) *lift to* Diff($M, \partial M$) *or* Homeo($M, \partial M$)*?*

Lifting to the universal cover $\tilde{S}$

We assume $\mathcal{L}$ exists and argue to a contradiction.

- $\mathcal{G} = \mathcal{L}(\Gamma)$ $\qquad$ $\mathcal{G}_1 = \mathcal{L}(\Gamma_1)$ $\qquad$ $f = \mathcal{L}(\alpha)$

- $\widetilde{M_2} =$ copy of the universal cover of M_2 in $\tilde{S}$.

- $\partial \widetilde{M_2} \subset \partial \tilde{S} = S^1_\infty$

- Each $g \in \mathcal{G}_1$ has a unique lift $\tilde{g} : \tilde{S} \to \tilde{S}$ that pointwise fixes $\partial \widetilde{M_2}$.

- For each interior fixed point of $A|M_2$, choose a lift $\tilde{f} : \tilde{S} \to \tilde{S}$ such that
  - $\partial\widetilde{M_2}$ is $\tilde{f}$-invariant.
  - $\tilde{f}|S_\infty$ has source-sink dynamics with at least two sources and two sinks .

Note: $\tilde{g}$ commutes with $\tilde{f}$ because

- g commutes with f
- $\tilde{g}|M_2$ commutes with $\tilde{f}|\partial\widetilde{M_2}$.

Global Fixed Points

Theorem 8. *Let* $\mathcal{G}$ *be a subgroup of* $\mathsf{Homeo}(D)$ *and let* f *be an element of the center of* $\mathcal{G}$*. Suppose* $K := \mathsf{Fix}(f) \cap \partial D$ *consists of a finite set with more than two elements each of which is either an attracting or repelling fixed point for* $f : D \to D$*. Let* $\mathcal{G}' \subset \mathcal{G}$ *denote the finite index subgroup whose elements pointwise fix* K*. Then* $\mathsf{Fix}(\mathcal{G}') \cap int(D)$ *is non-empty.*

Corollary 9. $\mathsf{Fix}(\mathcal{G}_1)$ *contains interior accumulation points.*

Thurston Stability Theorem

Theorem 10. (Thurston) *If G is a finitely generated non-trivial subgroup of* $\mathrm{Diff}(M^n)$ *and if there exists $x \in \mathrm{Fix}(G)$ such that $Dg_x = Id$ for all $g \in G$ then there is a non-trivial homomorphism from G to $\mathbb{R}$.*

There are no non-trivial homomorphisms from Γ_1 to $\mathbb{R}$.

The following corollary therefore completes the proof of Morita's Theorem.

Corollary 11. *If G is a finitely generated non-trivial subgroup of* $\mathsf{Diff}(M^2)$ *and if there exists an accumulation point $x \in \mathsf{Fix}(G)$ then there is a non-trivial homomorphism from G to $\mathbb{R}$.*

- $g \rightarrow \det(Dg_x)$ defines a homomorphism of G to $\mathbb{R}$.

- Dg_x has determinant one.

- Dg_x fixes a vector.

- $Dg_x = \begin{pmatrix} 1 & n_g \\ 0 & 1 \end{pmatrix}$.

- $g \rightarrow n_g$ defines a homomorphism from $\mathcal{G}$ to $\mathbb{Z}$.

- $n_g = 0$

Proving the Global Fixed Point Theorem

By doubling we can work in S^2 with invariant equator that has alternating attractors $\{p_i\}$ and repellers $\{q_i\}$.

$U :=$ basin of attraction for p_1 relative to f.

Exercise 12. *For all $g \in \mathcal{G}'$ there exists $n > 0$ so that U is the basin of attraction for p_1 relative to gf^n.*

Goal: Use the prime end compactification of U to find a $\mathcal{G}'$- fixed point not equal to p_i or q_i in the frontier of U.

Step 1 Find a $\mathcal{G}'$-fixed prime end $\widehat{y}$

- $\rho = \overrightarrow{p_1 q_1}$ lifts to $\widehat{\rho} = \overrightarrow{p_1 \widehat{w}}$ in D

- $\widehat{w}$ is a repeller for $\widehat{f}|\partial D$

- $\widehat{y} =$ endpoint of interval J of attraction

- This works equally well for gf^n so $\widehat{y}$ is $\mathcal{G}'$-fixed.

Step 2 $\widehat{y}$ corresponds to a $\mathcal{G}'$-fixed point y in the frontier of U

Step 3 $\widehat{y} \neq p_i, q_i$