

Geometric Makanin algorithm for solving equations in virtually free groups Notes taken by W. Malone

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Abstract

We give an algorithm deciding whether a system of equations and inequations in a virtually free group has a solution. This is done by giving a new geometric proof of Makanin's algorithm, based on Rips theory for foliated 2-complexes. Applications include the solution of equations in hyperbolic groups, and the isomorphism problem between hyperbolic group in presence of torsion. This is a joint work with Francois Dahmani.

Let G be a group then an equation in a group is a word in group elements (a_1, a_2, \dots) and variables (x_1, x_2, x_3) that is equal to the identity. One example would be

$$a_1 x_1 a_2 x_2 x_3^{-1} = 1.$$

Main Problem 1. *Given G and a system of equations and inequations can we decide if there exists a solution?*

In general the answer is no since this problem is akin to the conjugacy and word problems. If G is virtually abelian then the answer is yes. Romankov gave an example of a finitely generated nilpotent group in which there are no solutions. Another answer for a class of groups is given by the following theorem.

Theorem 1. *(Makanin's Algorithm) The answer to the main problem is yes when G is free.*

There have been many improvements and generalizations of this theorem to free products, graph products, and right angled Artin groups. In 2004 Plandovski gave an efficient (polynomial space) algorithm.

Theorem 2 (Dahmani-Guirardel, Lohney-Senizergues). *The answer to the main problem is yes when G is virtually free.*

Applications

Theorem 3. *(Dahmani-Guirardel, Rips-Sela, Dahmani) Yes for hyperbolic groups (with torsion)*

Theorem 4. *(Dahmani-Guirardel, Sela, Dahmani-Groves) Isomorphism problem for hyperbolic groups (with torsion)*

Our approach gives a new proof of solvability of equations in free group (IE new proof of Makanin's Algorithm)

1. Reduce to twisted equations in a free group.
2. Use Rips theory for foliated complexes.

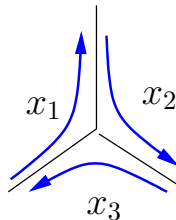
1.

Let $F \trianglelefteq G$ be finite index.

Equation $a_1x_1x_2x_3^{-1} = 1$

Twisted equation $a_1\varphi_1(x_1)a_2\varphi_2(x_2)\varphi_3(x_3^{-1}) = 1$ where $\varphi_1, \varphi_2, \varphi_3 \in \text{Aut}(F)$ are fixed nonvariable, and $\langle \varphi_1, \dots, \varphi_3 \rangle$ is finite in $\text{Out}(F)$. One additional requirement is that the φ_i need to preserve length.

For the rest of the talk we will assume no twisting and no inequations. Let's look at the triangular equation $x_1x_2x_3 = 1$. and encode this as a rose.

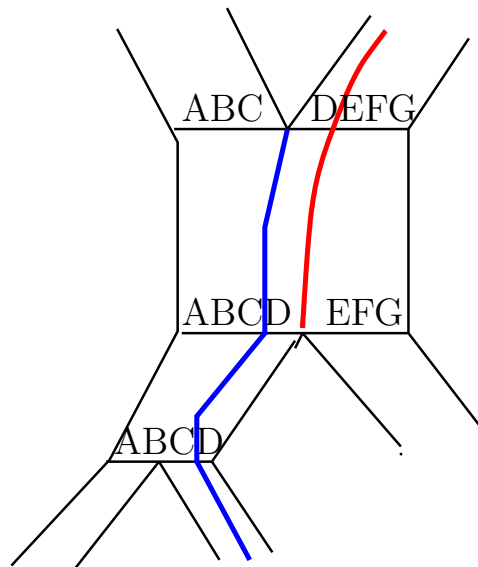
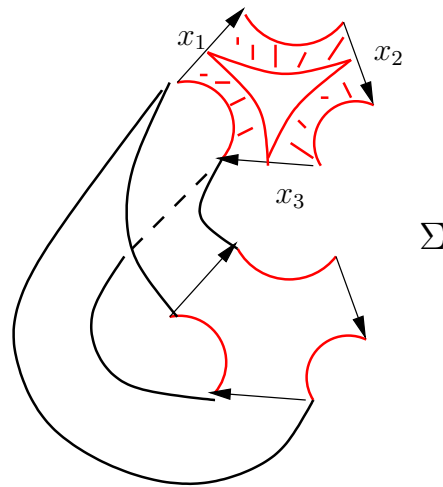


Starting from a system of equations we obtain a band complex. The solution of a band complex is a labeling compatible with the bands.

Definition 5. *Laminations induced by a solution σ of Σ .*

- *lamination = set of such colored segments*
- *extending lamination = making leaves longer*

After extending the leaves exist without reference to the labeling. All leaves will eventually become closed giving a complete lamination.



2. Algorithm

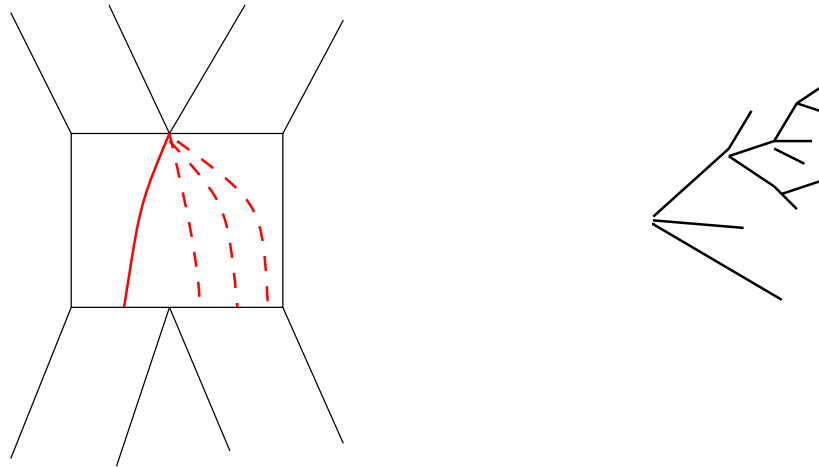
The idea of the algorithm is to try and find the shortest solution of Σ . There are two machines a lamination generator and a lamination analyzer.

Lamination Generator Produces all possible laminations and represents them as a rooted tree.

The tree stops at a complete lamination.

Lamination Analyzer (a) Looks for incompatibilities with constants (rejects the lamination if there is one IE an a on one end and a b on the other).

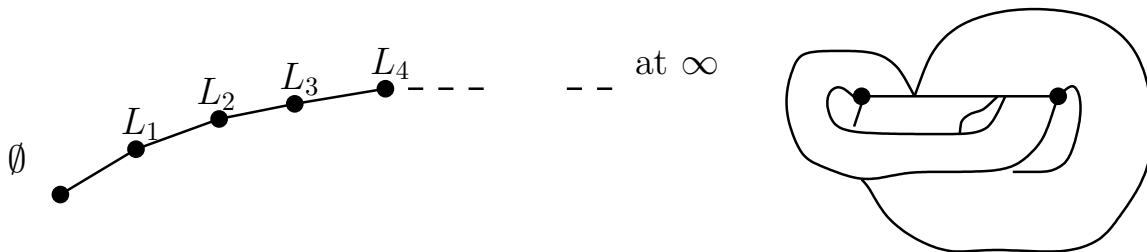
(b) If the lamination is complete and (a) is alright then a solution is found (STOP).



- (c) Is there an invariant “measure”? If no, then reject the lamination \mathcal{L} and reject everything to the right of it on the decision tree.
- (d) Try to produce a certificate proving that if σ induces \mathcal{L} then there exists a shorter solution $|\sigma'| < |\sigma|$. If yes then reject \mathcal{L} .
- STOP if every lamination is rejected.

This produces an algorithm.

Assume the algorithm does not stop.



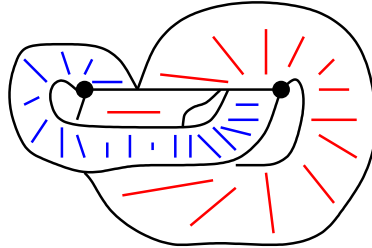
Facts:

- $L_1 \subset L_2 \subset L_3 \subset \dots \subset L_n \subset \mathcal{F}$ where \mathcal{F} is a complete foliation on Σ .
- \mathcal{F} has an invariant measure μ by part (c).

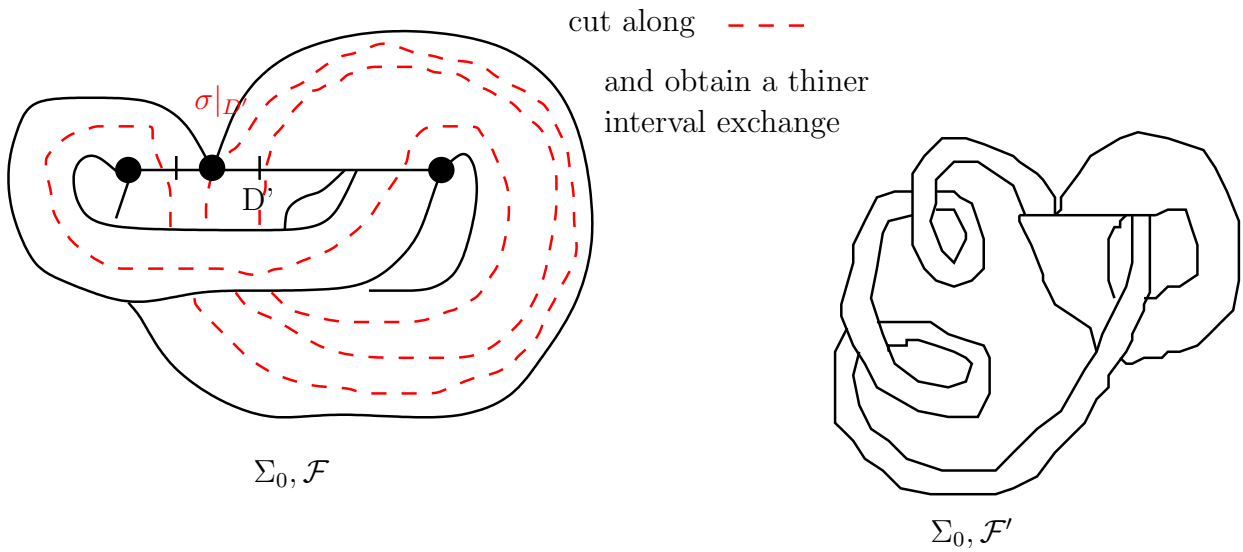
Can apply Rips theory (μ may fail to have full support and may have atoms, but for the talk this will be ignored). Rips theory implies that (Σ, \mathcal{F}) can have one of four types.

1. Simplicial leaves
2. Interval exchange type
3. Exotic
4. Homogeneous

Definition 6. *The interval exchange type implies that the finite set of operations coorespond to interval exchanges.*



Assume (Σ, \mathcal{F}) is an interval exchange.



Lemma 7. *For all ϵ there exists a certificate saying that if σ induces L_n then there exists σ' another solution with $|\sigma'| < \epsilon|\sigma|$.*