Coarse differentiation and the geometry of polycyclic groups.
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Abstract

By a theorem of Mostow, a group is polycyclic if and only if it is (up to finite groups) a lattice in a solvable Lie group. It is conjectured that the class of polycyclic groups is closed under the equivalence relation of quasi-isometry. I will discuss the geometry of these groups and recent progress toward the conjecture.

There are two main parts to the talk

1. Coarse Differentiation (Joint work with D. Fisher and K. Whyte)

2. Polycyclic groups (Irene Peng, D. Fisher)

The motivation behind coarse differentiation is the following theorem.

Theorem 1. *(Radamacher) Suppose $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ is lipschitz, then φ is differentiable almost everywhere.*

The proof uses the following theorem in the same vein.

Theorem 2. *(Lebesgue) If $\varphi : \mathbb{R} \to \mathbb{R}$ has bounded variation then φ is differentiable almost everywhere.*

1 Coarse Differentiation

Suppose that X and Y are geodesic metric spaces and $\varphi : X \to Y$ a (K,C) quasi-isometric embedding which is a map such that

$$\frac{1}{K} d_X(x_1, x_2) - C \leq d_y(\varphi(x_1), \varphi(x_2)) \leq K d_x(x_1, x_2) + C.$$

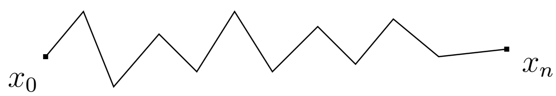
We would like a version of Radamacher's Theorem to hold in this situation. We can not differentiate because of the additive constant, so we might look to the asymptotic cone. More specifically we could look at $\widetilde{\varphi} : AC(X) \to AC(Y)$ which is a bilipschitz map. This approach works for $\mathbb{R}^n$ and whenever you have polynomial growth. Otherwise it is of no help. Thus we need to look elsewhere.

Definition 3. *$\varphi : X \to Y$ is affine if φ(geodesic) = geodesic.*

Definition 4. *Suppose $\epsilon > 0$ is given, a discrete path $x_0, x_1, \ldots, x_n$ is ϵ-efficient if*

$$d(x_0, x_n) \leq \sum d(x_i, x_{i+1}) \leq (1+\epsilon) d(x_0, x_n).$$

An intuitive way to think about this is that you have some wiggle room around a geodesic but not too much.



Definition 5. *$\gamma : [0, L] \to Y$ (think of as a quasi-geodesic) is ϵ-efficient on the scale r if when we let ($\Delta(\gamma, r) = \sum_{i=0}^{n-1} d(\gamma(\frac{(i+1)r}{2}), \gamma(\frac{ir}{L}))$ which we think of as the coarse length of γ on the scale r)*

$$\Delta(\gamma, r) \leq (1+\epsilon) d(\gamma(0), \gamma(L)) \leq (1+\epsilon)\Delta(\gamma, L)$$

Definition 6. *$\varphi : X \to Y$ is ϵ-affine on scale r, if φ(geodesic) is ϵ-efficient on scale r.*

Lemma 7. *(Coarse Differentiation) Suppose $\varphi : X \to Y$ is a (K,C) quasi-isometric embedding. Let $\mathcal{F}$ be a family of geodesics in X. For any $\epsilon > 0$ and any $\theta > 0$ there exist constants $C \ll r \ll R \ll L_0$ such that the following holds. Suppose $L > L_0$ and $B \subset X$ which we think of as a box of size L. Let*

$$\mathcal{F}_{B,L} = \{\gamma \cap B | \gamma \in \mathcal{F} \text{ and } \frac{L}{100} \leq |\gamma \cap B| < 100L\}$$

and assume that $\gamma \cap B$ is connected. Let $\mathcal{F}'$ be a collection of geodesic segments obtained by subdividing $\gamma \in \mathcal{F}_{B,L}$ into segments of length R. Then for at least $(1-\theta)$ fraction of segments (ratio of segments that satisfy the conditions of $\mathcal{F}'$ to those that do not is $(1-\theta)$) $\gamma \in \mathcal{F}'$, $\varphi(\gamma')$ is ϵ-efficient on scale r.

The picture that you should have in mind is the following. Take your geodesics and break them into segments that are mapped to efficient paths in the range. Or morally you can consider the lemma as stating that $\varphi|_B$ is ϵ-affine on scale r.

Proof. (Of Lemma) Pick $C \ll r_0 \ll r_1 in \ll \cdots \ll r_m$ where the number of scales can be as large as you want and require that

$$\frac{r_{i+1}}{r_i}\mathbb{N}$$

. Let $L_0 \gg r_m$. Suppose $\gamma \in \mathcal{F}_{B,L}$. Given m, subdivide γ into segments of length r. Let $\delta_m(\gamma)$ be the fraction of these segments for which the image under φ is not

ϵ-efficient on the scale r_{m-1}. Ideally we would like $\delta_m(\gamma)$ (which is a number between 0 and 1) to be as close to 0 as possible.

Suppose I is one of these segments. If $\varphi(\gamma|_I)$ is not ϵ-efficient, then

$$\Delta(\gamma|_I, r_{m-1}) \geq \Delta(\gamma|_I, r_m) + \epsilon\frac{|I|}{|2k|}.$$

Via the triangle inequality we know that

$$\Delta(\gamma_I, \gamma_{m-1}) \geq \Delta(\gamma|I, r_m).$$

Adding over all segments we obtain

$$\Delta(\gamma, r_{m-1}) \geq \Delta(\gamma, r_m) + \frac{\epsilon|\gamma|}{2k}\delta_m(\gamma).$$

Now apply to every m to get

$$\Delta(\gamma, r_0) \geq \Delta(\gamma, r_m) + \frac{\epsilon|\gamma|}{2k}\sum_{m=1}^{M}\delta_m(\gamma).$$

This shows

$$2k|\gamma| \geq \frac{\epsilon}{2k}|\gamma|\sum_{m=1}^{M}\delta_m(\gamma).$$

And thus

$$\sum_{m=1}^{M}\delta_m(\gamma) \leq \frac{4k^2}{\epsilon}$$

by choosing M large we can make $\delta_m(\gamma) < \theta$ for some m. Average over all geodesics crossing the box to get

$$\sum_m = 1^M\left(\frac{1}{|\mathcal{F}_{B,L}|}\sum_{\gamma\in\mathcal{F}_{B,L}}\delta_m(\gamma)\right) \leq \frac{4k^2}{\epsilon}$$

where the quantity in parenthesis is less than θ. □

This is similar to the proof of 2 although scales get bigger not smaller and hence you cannot pass to the limit.

2 Polycyclic groups

Definition 8. Γ *polycyclic means that you can find*

$$\Gamma_0 = \{e\} \trianglelefteq \Gamma_1 \trianglelefteq \ldots \trianglelefteq \Gamma_n = \Gamma$$

such that Γ_{i+1}/Γ_i *is cyclic.*

Theorem 9. *(Mostow)* Γ *is polycyclic if and only if* Γ *is a lattice in a solvable unimodular lie group* G.

An important note is that solvable lie groups are not necessarily $CAT(0)$.

Conjecture 1. *(QI rigidity of Polycyclic groups) Suppose* Γ *is a polycyclic group and* Λ *is a group quasi-isometric to* Γ*, then* Λ *is virtually polycyclic.*

Osin defined the exponential radical to be

$$R_{exp}G = \{g | d(g^n, e) \leq o(log(n))\}.$$

Two important facts are that $R_{exp}G \trianglelefteq G$ and the quotient $G/R_{exp}G$ has polynomial growth.

Conjecture 2. *(Generalized Farb-Mosher conjecture) If* $\varphi : G \to G$ *is a quasi-isometry then* φ *preserves the orbits of* $R_{exp}G$.

Theorem 10. *(Peng) If* $G = \mathbb{R}^k \ltimes \mathbb{R}^n$ *where* $\mathbb{R}^n = R_{exp}G$ *then the Farb-Mosher conjecture is true as well as quasi-isometric rigidity. For* $k = 1$ *and* $n = 2$ *is the case of Sol geometry which was done by Eskin-Fisher-Whyte.*

In the general case

$$1 \to R_{exp}G \to G \to G/R_{exp}G \to 1$$

which does not split.