The Geometry of Hyperbolic Manifolds via the Curve Complex

Talk by Ken Bromberg

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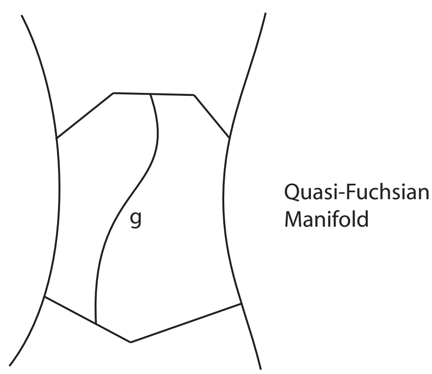
One of the goals of this course has been to demonstrate how the combinatorics of the curve complex relate to the geometry of quasi-Fuchsian 3-manifolds. In this lecture we will see another example of this, which employs the Lipschitz surfaces that were introduced earlier.

For $X, Y \in Teich(S)$, let $Q(X, Y)$ denote the quasi-Fuchsian manifold with conformal boundaries X and Y.

We want to describe the relationship between the size of the convex core of $Q(X, Y)$ and the distance between points in the curve complex associated to X and Y. Let g be the shortest geodesic connecting the convex core boundaries.



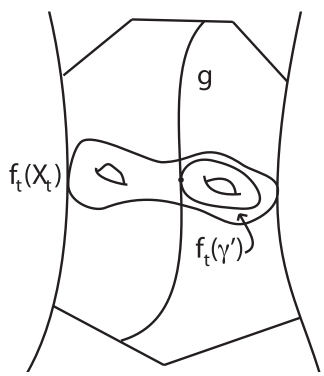
We associate points in the curve complex to X and Y as follows. Both X and Y have some closed curve γ of length less than B_S, the Bers constant for S. Let α be such a curve on X and β be such a curve for Y. We define $d_{\mathcal{C}}(X, Y)$ to be $d_{\mathcal{C}}(\alpha, \beta)$, the distance in the curve complex between α and β. As α and β are not unique, this is not well defined. However, it is an exercise to check that given any L there is a constant N_L such that if γ_1 and γ_2 satisfy $l_X(\gamma_i) \leq L$, then $d_{\mathcal{C}}(\gamma_1, \gamma_2) \leq N_L$. This shows that $d_{\mathcal{C}}(X, Y)$ is defined up to some constant N_{B_S}. The first goal of this talk will be to prove the following theorem.

THEOREM 1: (Brock-Bromberg) There exist constants K_1 and K_2 depending only on the topology of S such that $l(g) \geq K_1 d_{\mathcal{C}}(X, Y) - K_2$.

Proof. By Sullivan's theorem, the convex core boundaries are close in Teichmüller space to the conformal boundaries. Thus as α and β have bounded length on X and Y respectively, they have bounded length on the convex core boundaries. The length of these curves in the 3-manifold is smaller than their lengths on the convex core boundaries, so we can find uniform bounds for the geodesic representatives of α and β in the 3-manifold. This can also be given directly by the following inequality of Bers relating to extremal length: if α^* and β^* are the geodesic representatives of α and β in $Q(X, Y)$, then $l_Q(\alpha^*) \leq 2B_S$.

Let $f_t : X_t \to Q(X, Y)$ be a 1-Lipschitz interpolation as described in the previous lecture, with $\alpha^* \subset f_0(X_0)$, $\beta^* \subset f_1(X_1)$ and $(X_t, f_t) \in LIP(Q(X, Y))$. We have, by the discussion from last time, that $f_t(X_t)$ is contained in the convex core. As f_t is a homotopy whose image is in the convex core for all t, $f_t(X_t)$ must separate the convex core by Waldhausen's theorem. As g goes from one convex core boundary to the other, $f_t(X_t) \cap g \neq \varnothing$ for all t.

For γ a simple closed curve on S, let $I(\gamma) \subset [0, 1]$ be all t for which there is a loop γ' on X_t which is homotopic to γ and satisfies $l_{X_t}(\gamma') < B_S$ and $f_t(\gamma') \cap g \neq \varnothing$.



The union over all γ of the $I(\gamma)$ is the whole interval, as any point on a hyperbolic surface as a loop through it of length less than B_S.

An important fact that we will need is that there exists a number N such that if $I(\gamma_1) \cap I(\gamma_2) \neq \varnothing$ then the distance in the curve complex between γ_1 and γ_2 is less than N. This follows from the collar lemma, as a curve that crosses a bounded curve many times cannot itself be bounded.

Now we want to adjust the definition of the maps f_0 and f_1 so that not only are these maps in $LIP(Q(X, Y))$, but that also the length of α on X_0 is exactly the length of α^*, and the length of β on X_1 is exactly the length of β^*. Recall that using the triangulation construction from the previous lecture this is true for the surfaces with cone points, but on the hyperbolic surfaces in the conformal class of these cone surfaces the length may be shorter (though it can't be longer by a lemma of Ahlfors, as was mentioned last time). It is possible to find such surfaces, though we won't go in to the details here. For now, let's assume that f_0 and f_1 have been chosen so that this is the case.

Suppose $0 \in I(\gamma)$. Then γ and α are bounded on the same surface, so $d_{\mathcal{C}}(\alpha, \gamma) < N$, and likewise if $1 \in I(\gamma)$ then $d_{\mathcal{C}}(\beta, \gamma) < N$. Using a finite cover of the interval from the set $\{I(\gamma\}$'s, we can therefore construct a course path in the curve complex from α to β, that is we get a sequence $\alpha = \gamma_1, \gamma_2, ..., \gamma_{n+1} = \beta$ where $d_{\mathcal{C}}(\gamma_i, \gamma_{i+1}) < N$ and $I(\gamma_i) \neq \varnothing$ for $1 \leq i \leq n$.

This says that if α and β are far apart, then we can find a large number of distinct homotopy classes of curves of bounded length that go through the geodesic g. More precisely, the number of distinct homotopy classes of curves with length less or equal to $2B_S$ is greater than n, the number of curves in the path above. Furthermore, we know that $d_{\mathcal{C}}(\alpha, \beta) \leq (n+1)N$, so $n \geq \frac{1}{N} d_{\mathcal{C}}(\alpha, \beta) - 1$.

The following Lemma is an easy corollary of the Margulis Lemma.

Lemma: For all L, there exists a $D = D(L)$ such that there are at most D distinct homotopy classes of curves with length at most L intersecting any set of diameter less than 1.

Exercise: Prove the preceding lemma.

From this fact we see that if $D = D(2B_s)$ then there are at most D curves in any segment of length 1 on the geodesic g. From this we have that $Dl(g) \geq n$, and combining this with the previous inequality we see that $l(g) \geq \frac{1}{ND} d_{\mathcal{C}}(\alpha, \beta) - \frac{1}{D}$.

□

The following theorem gives a similar result that allows us to control distances between geodesics in the three manifold.

Theorem 2: Let $M = \mathbb{H}^3/\Gamma$, $\Gamma \cong \pi_1(S)$. Given $L > 0$, there exist constants K_1 and K_2 depending on L and S such that if α and β are simple closed curves whose length in the three manifold is bounded above by L, then the distance in the manifold between geodesic representatives α^* and β^* satisfies $d_M(\alpha^*, \beta^*) \geq K_1 d_{\mathcal{C}}(\alpha, \beta) - K_2$.

The setup of the proof of this theorem is exactly the same. Take a geodesic g from α^* to β^* and look at a 1-Lipschitz homotopy between surfaces containing these geodesics. The proof of theorem 1 does not go through directly, as in that setting we knew that the image of our homotopy stayed in the convex core, and therefore always had to intersect the geodesic between the convex core boundaries. Now the image of our homotopy may not always intersect g as it need not stay between α^* and β^*. By a geometric limit argument, however, one can get around this problem.

We will now give some applications of this result. For $M \cong S \times \mathbb{R}$, let $\mathcal{C}(M; L)$ denote the collection of all simple geodesics in M of length less or equal to the L, where $L \geq B_S$. Here simple means homotopic to a simple curve in one of the surface fibers.

THEOREM 3: (Minsky) $\mathcal{C}(M; L)$ is K-quasiconvex in $\mathcal{C}(S)$, where K depends only on S.

This theorem may seem trivial in a quasi-Fuchsian manifold, as the convex core is compact and therefore there are only finitely many homotopy classes of curves of length less than L. Any finite set is quasi-convex, so this may not seem to hold much information. Because the K in the theorem depends only on topology, however, this does give interesting information even in this case. In the case of degenerate groups, when the convex core is the whole manifold, proving quasi-convexity even for a nonuniform K is nontrivial, so here this theorem is even more useful.

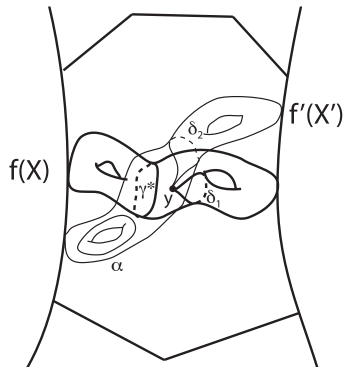
Proof. Let $\mathcal{C}_0$ denote the vertices of the curve complex and let $\mathcal{S}(C_0(S))$ denote the subsets of the curve complex. Define $\Pi_M : \mathcal{C}_0 \to \mathcal{S}(\mathcal{C}_0(S)$ to be the map given by the conditions $\alpha \in \Pi_M(\gamma)$ if there is $(X, f) \in LIP(M)$ with $f(X) \cap \gamma^* \neq \varnothing$ and $l_X(\alpha) \leq L$.

LEMMA: Π_M is a coarse Lipschitz projection, i.e.
1. There exists D_1 such that $\text{diam}(\Pi_M(\gamma)) \leq D_1$
2. There exists D_2 such that $d_{\mathcal{C}}(\gamma_1, \gamma_2) = 1$ implies $\text{diam}(\Pi_M(\gamma_1) \cup \Pi_M(\gamma_2)) \leq D_2$
3. If $\gamma \in \mathcal{C}(M; L)$ then $\gamma \in \Pi_M(\gamma)$

Property 3 follows immediately from the ability to construct 1-Lipschitz surfaces on which map a given geodesic on the surface to a geodesic in the three manifold of the same length. Given any γ, there exists $(X, f) \in LIP(M)$ such that $l_X(\gamma) = l_M(\gamma)$, so the surface intersects the geodesic γ^* in the three manifold.

Property 2 can be derived from property 1. Given γ_1 and γ_2 with $d_{\mathcal{C}}(\gamma_1, \gamma_2) = 1$, then there exists $(X, f) \in LIP(M)$ with $f(X) \cap \gamma_i \neq \varnothing$ for $i = 1, 2$ (in fact we can find such a map with $l_X(\gamma_i) = l_M(\gamma_i)$ for $i = 1, 2$). This shows that $\Pi_M(\gamma_1) \cap \Pi_M(\gamma_2) \neq \varnothing$, thus $\text{diam}(\Pi_M(\gamma_1) \cup \Pi_M(\gamma_2)) \leq \text{diam}(\Pi_M(\gamma_1)) + \text{diam}(\Pi_M(\gamma_2)) \leq 2D_1$.

So we are left with proving property 1. To do this we will use Theorem 2 above. Suppose α and β are two curves in $\Pi_M(\gamma)$. We can find a $(X, f) \in LIP(M)$ with $\gamma^* \subset f(X)$. As $\alpha \in \Pi_M(\gamma)$, there exists another surface (X', f') intersecting γ^* on which α is short. Let $y \in f'(X') \cap f(X)$. There are curves δ_1 on $f(X)$ and δ_2 on $f'(X')$ of length less than the Bers constant through y.



These curves intersect, so $d_M(\delta_1, \delta_2) = 0$. For simplicity, assume that δ_1 and δ_2 are geodesics in the manifold, so that theorem 2 applies. Applying Theorem 2 we get that $d_{\mathcal{C}}(\delta_1, \delta_2) \leq \frac{K_2}{K_1}$. α and δ_1 are bounded curves on the same surface [how do we use Theorem 2? It only seems to apply to geodesics], so we have a bound D to their distance in the curve complex, and likewise $d_{\mathcal{C}}(\delta_2, \gamma) \leq D$, so we get that $d_{\mathcal{C}}(\alpha, \gamma) \leq 2D + \frac{K_2}{K_1}$. We can now apply the same argument to β to get that $d_{\mathcal{C}}(\beta, \gamma) \leq 2D + \frac{K_2}{K_1}$, so $d_{\mathcal{C}}(\beta, \alpha) \leq 4D + 2\frac{K_2}{K_1}$.

The following Lemma, whose proof will not be given, finishes the proof of Minsky's theorem.

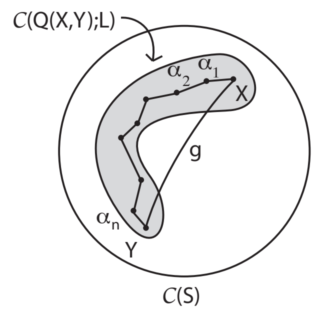
Lemma: The image of a course Lipschitz projection Π from a δ-hyperbolic space to itself is quasiconvex, where the quasiconvexity constants depend only on the constants for Π.

□

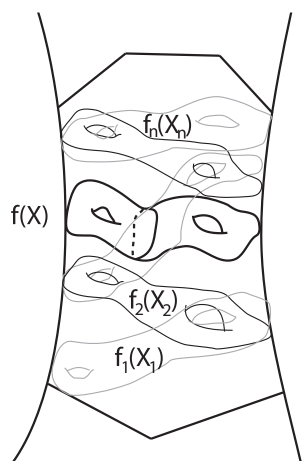
The following theorem shows how quasi-convexity can be used to show that a neighborhood of a curve complex geodesic contains all bounded length curves in a quasi-Fuchsian manifold.

Theorem 4: If g is a geodesic in the curve complex connecting a bounded length curve on $X \in Teich(S)$ to a bounded length curve on $Y \in Teich(S)$, then there exists K such that $\mathcal{C}(Q(X,Y); L) \subset \mathcal{N}_K(g)$, where K depends on S and L.

Proof. Let g be a path in the curve complex from X to Y (where by X and Y we mean any short curve on these structures). By Theorem 3, $\mathcal{C}(Q(X,Y);L)$ is quasi-convex, so we can find some path $P \subset \mathcal{C}(Q(X,Y);L)$ that stays a bounded distance from g.



Let $\gamma \in \mathcal{C}(Q(X,Y);L)$, and let $(X,f) \in LIP(M)$ be such that $\gamma \subset f(X)$. Let $\alpha_1, \alpha_2, ..., \alpha_n$ be the points along P, so that $d_{\mathcal{C}}(\alpha_i, \alpha_{i+1}) = 1$. We can find (X_i, f_i) such that α_i and α_{i+1} are simultaneously bounded on $f_i(X_i)$.



This ladder of surfaces eventually intersects $f(X)$, and applying the first lemma from the proof of theorem 3, we see that γ must be some uniform distance K' from some curve α_k in the path P. This shows that $\mathcal{C}(Q(X,Y);L) \subset \mathcal{N}_{K'}(P)$, so as P is uniformly close to g, there exists K such that $\mathcal{C}(Q(X,Y);L) \subset \mathcal{N}_K(g)$.

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