Solutions for Section 9.3

2. Find all real solutions for the differential equation $\frac{dx}{dt} + 3x = 7$.

The homogeneous system $\frac{dx}{dt} + 3x = 0$, or $\frac{dx}{dt} = -3x$, has general solution $x(t) = Ce^{-3t}$, since this is just a growth/decay equation.

By inspection, we notice that $x_p(t) = \frac{7}{3}$ is a particular solution to the original system.

Hence, the complete solution to this differential equation is

$$x(t) = Ce^{-3t} + \frac{7}{3},$$

where C is any real constant. □

4. Find all real solutions to the differential equation $\frac{dx}{dt} - 2x = \cos 3t$.

Again, the general solution to the homogeneous equation $\frac{dx}{dt} - 2x = 0$ is just $x(t) = Ce^{2t}$.

Using Fact 9.3.10, we guess that we can find a particular solution with the form $x_p(t) = P\cos 3t + Q\cos 3t$. To solve for P and Q, we substitute into the original equation to get

$$-3P\sin 3t + 3Q\cos 3t - 2P\cos 3t - 2Q\sin 3t = \cos 3t.$$

But this must be true for all t; in particular, substituting $t = 0$ and $t = \frac{\pi}{6}$, we get the equations $-2P + 3Q = 1$ and $-3P - 2Q$, which have solution $P = -\frac{2}{13}$, $Q = \frac{3}{13}$. Finally, we conclude that the general solution of the original differential equation is

$$x(t) = Ce^{2t} - \frac{2}{13}\cos 3t + \frac{3}{13}\sin 3t,$$

where C is any real constant. □

8. Find all real solutions to the differential equation $\frac{d^2x}{dt^2} + 3\frac{dx}{dt} - 10x = 0$.

Since this equation is homogeneous of degree two, its solutions form a two-dimensional linear space. To find a basis for this space, observe the characteristic polynomial: $p_T(\lambda) = \lambda^2 + 3\lambda - 10 = (\lambda + 5)(\lambda - 2)$. Hence $\{e^{-5t}, e^{2t}\}$ is a basis for the solution space, and the general solution of the differential equation is

$$x(t) = c_1e^{-5t} + c_2e^{2t},$$

where c_1, c_2 are any real constants. □

24. Solve the initial value problem $\frac{dx}{dt} + 3x = 7$, $x(0) = 0$.

From problem 2, we know that the general solution of this equation is $x(t) = Ce^{-3t} + \frac{7}{3}$. Now, we know that $x(0) = 0$; but we also know that $x(0) = Ce^0 + \frac{7}{3} = C + \frac{7}{3}$. This tells us that $C = -\frac{7}{3}$, and so the solution to the initial value problem is

$$x(t) = -\frac{7}{3}e^{-3t} + \frac{7}{3}.$$

□

28. Solve the initial value problem $f''(t)+f'(t)-12f(t) = 0$, $f(0) = f'(0) = 0$.

Since this differential equation is homogeneous, we factor the characteristic polynomial, $p_T(\lambda) = \lambda^2 + \lambda - 12 = (\lambda + 4)(\lambda - 3)$, to see that the general solution is $f(t) = c_1e^{-4t} + c_2e^{3t}$. Now substituting in the initial conditions, we get

$$\begin{aligned} c_1 + c_2 &= f(0) = 0 \\ -4c_1 + 3c_2 &= f'(0) = 0, \end{aligned}$$

which tells us that $c_1 = c_2 = 0$. Thus, the solution to the initial value problem is $f(t) = 0$. □