Mathematics 420, Spring 2002

On Exact ODEs and Integrating Factors

- Let $(t,x) \in Q \subset R^2$. Given an ODE of the form $M(t,x)x' + L(t,x) = 0$ one can solve it as an *exact DE* if there exists some *exact integral function* $F(t,x)$ s.t. $L = \frac{\partial F}{\partial t}$ and $M = \frac{\partial F}{\partial x}$. In that case, any solution of the ODE $u(t)$ will also satisfy $F(t,u(t)) = C$ for some constant C.

  Extra Credit Problem 1: Use implicit differentiation to demonstrate that any $\gamma(t)$ defined on some interval I and satisfying $F(t,\gamma(t)) = C$ will be a solution of the original ODE. Prove that $M(t,\gamma(t)) \neq 0$ for all interior points of the interval I.

- The equality of mixed partials implies that there is no hope of finding such a (continuously differentiable) F if $M_t \neq L_x$. As shown in the lecture and in the book, if $M_t = L_x$ and Q is a rectangle, then such an F always exists and is defined as

$$F(t,x) = \int_{t_0}^{t} L(\tau, x_0)\, d\tau + \int_{x_0}^{x} M(t,y)\, dy.$$

  More generally, an *exact integral* F always exists on Q provided $M_t = L_x$ and Q is *simply connected*. (A set is simply connected if "there are no holes in it", i.e. every closed curve in Q can be continuously "shrunk" to a point that also lies in Q.)

  Extra Credit Problem 2: Suppose Q is not simply connected. Prove that it's possible that no such F exists even if $M_t = L_x$ everywhere on Q.

- If the ODE is given in the canonical form $x' = f(t,x) = \frac{-L(t,x)}{M(t,x)}$, the slope of the solution is infinite (i.e. the solution is undefined) at (t_1,x_1) if $M(t_1,x_1) = 0$. Nevertheless, it is possible that such a point belongs to a level set of F. (Numerous examples in the textbook.)

  Extra Credit Problem 3: Given that the ODE is exact, prove the uniqueness of the solution passing through every point where $M \neq 0$.

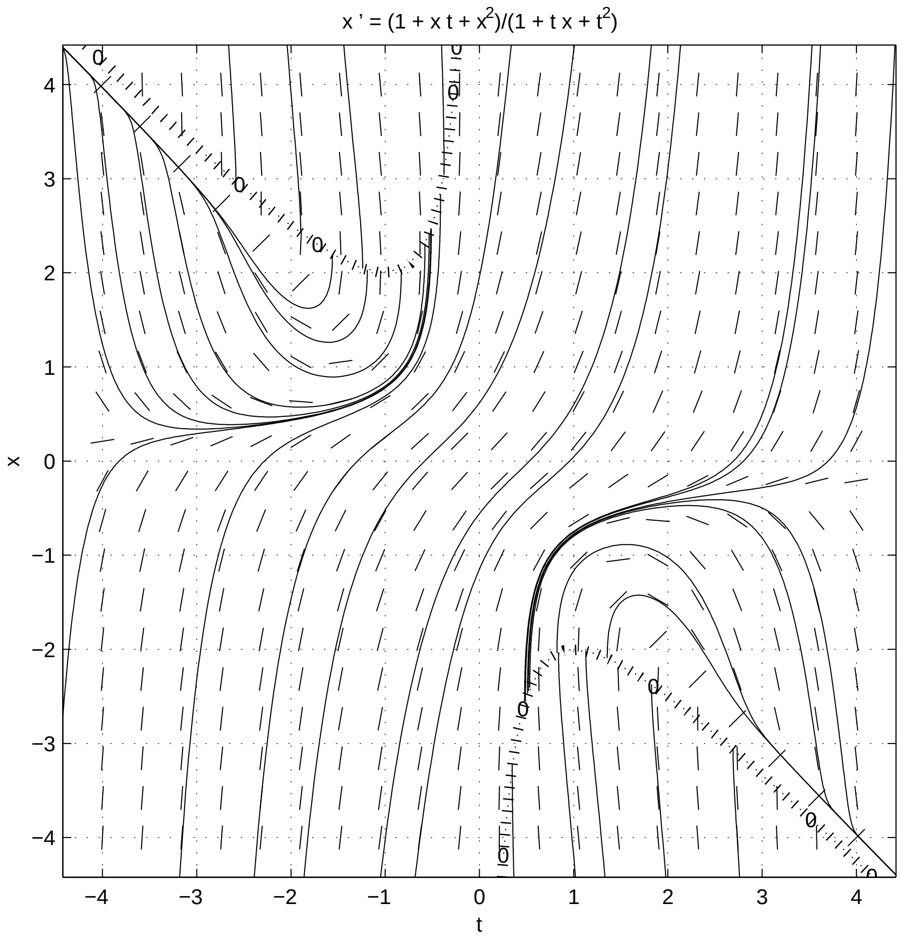
- An *integrating factor* $\mu(t,x)$ might be useful when the ODE is **not** exact, but could become exact after multiplication by a non-zero function μ, i.e. if it is possible to find F and μ such that

$$\mu L = \frac{\partial F}{\partial t} \qquad \text{and } \mu M = \frac{\partial F}{\partial x}.$$

Consider, for example, a non-linear non-separable ODE

$$x' = -\frac{1+tx+x^2}{1+tx+t^2},$$

defined everywhere on R^2 except for two curves where $1+tx+t^2 = 0$. As you can see in the below figure, the solutions are defined and unique everywhere except on those curves. (Note the potential applications for funnels/antifunnels here.)



Extra Credit Problem 4: Show that the above ODE is not exact ($M_t \neq L_x$). Use the integrating factor $\mu(t,x) = e^{xt}$ and show the equality of the new mixed partial derivatives ($(\mu M)_t = (\mu L)_x$). Find the corresponding F.

Extra Credit Problem 5: Find an example of a problem, where the needed integrating factor $\mu(t,x)$ is zero at some points in Q. What can you say about the solutions of the original ODE passing through those points?

Solutions:

Extra Credit Problem 1: As it is formulated, the problem is not well-posed; two additional assumptions are needed to make the statement meaningful[1].

Assumption 1: $F(t,x)$ and $\gamma(t)$ are continuously differentiable everywhere.

This is needed to use the chain rule, to assert the continuity of M and L, **and** to claim that the slope of $\gamma(t)$ is finite inside I.

Given the above,

$$\frac{d}{dt}\left[F(t,\gamma(t))\right] = F_t(t,\gamma(t)) + F_x(t,\gamma(t))\gamma'(t) = L(t,\gamma(t)) + M(t,\gamma(t))\gamma'(t) = 0,$$

therefore $\gamma(t)$ is a solution to the ODE.

Since $\gamma(t)$ is continuously differentiable, $\gamma'(t)$ is well-defined (and finite[2]) on I. For this to imply that $M(t,\gamma(t)) \neq 0$ we need an additional

Assumption 2: $L(t,\gamma(t)) \neq 0$. More generally, if the ODE was given in the canonical form, then the slope at the point (t_0,x_0) becomes infinite iff

$$lim_{(t,x)\to(t_0,x_0)} \frac{M(t,x)}{L(t,x)} = 0.$$

Thus, $M(t_0,x_0) = 0$ is not a threat in itself - as long as L has a zero of even higher order at (t_0,x_0).

Extra Credit Problem 2: This problem should have been reminiscent of the sufficient conditions for the vector field to be conservative. Here, however, we are asking why such conditions are also necessary. Consider the simplest possible *not simply-connected* $Q = R^2 \backslash (0,0)$. The two functions

$$M(t,x) = \frac{t}{x^2+t^2}; \qquad L(t,x) = \frac{-x}{x^2+t^2}$$

are continuously differentiable everywhere (except at the origin). Moreover, $M_t = L_x$ on Q (check!). We will assume the existence of a continuously differentiable $F(t,x)$ such that $F_t = L$ and $F_x = M$, just to arrive at a contradiction later.

Consider any closed simple smooth curve in Q parameterized by a periodic function $g(s)$ (i.e., $g(s) = (g_1(s), g_2(s)) \in Q$ and $g(s_1) = g(s_2)$ iff $s_1 - s_2 = nT$, where T is the period of g and n is some integer). Note that

$$\frac{d}{ds}\left[F(g(s))\right] = \nabla F(g(s)) \cdot g'(s) = F_t(g(s)){g_1}'(s) + F_x(g(s)){g_2}'(s).$$

---

[1] Sorry. I guess, nobody will get full credit for this one - including the instructor.

[2] If the interval I is open, the derivative **can** be infinite at the end-points.

Thus, by the Fundamental Theorem of Calculus,

$$\int_{s_1}^{s_1+T} \left[L(g(s)){g_1}'(s)+M(g(s)){g_2}'(s)\right]\,ds = \int_{s_1}^{s_1+T}\left[\frac{d}{ds}F(g(s))\right]\,ds = F(g(s_1+T))-F(g(s_1)) = 0.$$

We will now consider a particular parameterization of the unit circle $g(s) = (\cos(s),\sin(s))$ and proceed to compute the above integral explicitly:

$$\int_0^{2\pi}\left[\frac{-\sin(s)}{\cos^2(s)+\sin^2(s)}(-\sin(s))+\frac{\cos(s)}{\cos^2(s)+\sin^2(s)}(\cos(s))\right]\,ds = \int_0^{2\pi} ds = 2\pi,$$

which is a contradiction.
NB: It is actually possible to define F even in cases like the one above; however, F will have to be a strange object - *a multi-valued function*. See Problem #2.8.3 in "Ordinary Differential Equations" by I.G. Petrovski for details.

Extra Credit Problem 3: This follows directly from the Implicit Function Theorem applied to F at that point. The unique (local) representation of x as a function of t is guaranteed to exist in the neighborhood of every point where $F_x = M \neq 0$.
Incidentally, the theorem says **nothing** about the points where $M = 0$. Indeed, it can still be that the smooth representation $x = \gamma(t)$ is unique there. Or it can be non-unique and smooth. Or even non-unique and non-differentiable. (Can you find some examples?)

Extra Credit Problem 4: Let $L = 1+tx+x^2$ and $M = 1+tx+t^2$. First,

$$M_t = x+2t \neq t+2x = L_x,$$

and therefore the ODE is not exact. Secondly, for $\mu(t,x) = e^{xt}$,

$$(\mu M)_t = xe^{xt}(1+tx+t^2)+e^{xt}(x+2t) = (x+t)(2+tx)e^{xt};$$

$$(\mu L)_x = te^{xt}(1+tx+x^2)+e^{xt}(t+2x) = (x+t)(2+tx)e^{xt};$$

thus, it truly is an integrating factor. Finally,

$$F(t,x) = \int(\mu M)\,dx+\phi_1(t) = (t+x)e^{xt}+\phi_1(t),$$

$$F(t,x) = \int(\mu L)\,dt+\phi_2(x) = (t+x)e^{xt}+\phi_2(t),$$

and we can take $F(t,x) = (t+x)e^{xt}$.

Extra Credit Problem 5: To use a really simple example, consider the non-exact ODE $2tx'+x = 0$, for which $\mu(t,x) = x$ is an integrating factor (check!). Note that $M(t,x) = 2t$

is non-zero everywhere except on the x-axis; yet (μM) as also zero on t-axis. Did we manage to introduce new (irrelevant) points where the slope is infinite? The answer is no, and the reason is stated in Problem 1: the slope is infinite (and, therefore, no solution to ODE exists) only at the points where

$$lim_{(t,x)\to(t_0,x_0)}\frac{M(t,x)}{L(t,x)} = 0.$$

Since we are multiplying by μ bot M and L, the limit in question does not change even if μ is zero at that point.