1. Decide whether or not the following sets with given binary operations are groups. Justify your answer.
    a. \( G_1 = \mathbb{Z} \) with \( a \ast b = \max(a, b) \).
    b. \( G_2 = \mathbb{Z} \) with \( a \ast b = a - b \).
    c. \( G_3 = \mathbb{R}^+ \) with \( a \ast b = a \cdot b \).
    d. \( G_4 = \mathbb{Q} \) with \( a \ast b = a \cdot b \).

2. Let \( G \) be a group. Show that \( G \) is abelian if and only if \( (a \cdot b)^{-1} = a^{-1} \cdot b^{-1} \) for every \( a, b \in G \).

3. Show that if \( G \) is a finite group with an even number of elements, there must be some \( a \in G \) with \( a \neq e \) and \( a^2 = e \).

4. If \( G \) is a group, and \( H \) and \( K \) subgroups, then are \( H \cup K \) and \( H \cap K \) also subgroups?

5. Which of the following functions are homomorphisms? If a function is a homomorphism, please compute its kernel.
    a. \( \phi : (\mathbb{R}^\times, \cdot) \to (\mathbb{C}^\times, \cdot) \) defined by \( \phi(x) = x + ix \).
    b. \( \psi : (M_2(\mathbb{R}), +) \to (\mathbb{R}, +) \) defined by \( \psi \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = a + d \).
    c. \( \pi : (\mathbb{C}^\times, \cdot) \to (\mathbb{R}^\times, \cdot) \) defined by \( \pi(a + bi) = a^2 + b^2 \).
    d. \( \rho : (\mathbb{R}, +) \to (\text{GL}_2(\mathbb{R}), \cdot) \) defined by \( \rho(a) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \).