1. Classify the equilibrium points of the system:

\[
\begin{align*}
\dot{x} &= -x + x^3y \\
\dot{y} &= y - x^2
\end{align*}
\]

Solution: The equilibrium points of the system satisfy \( y = x^2 \) and \(-x + yx^3 = 0\), i.e. \(-x + x^5 = 0\), which has the (real) roots \( x = 0, \pm 1 \). I.e., the equilibrium points are \((0, 0), (-1, 1), (1, 1)\). The Jacobian is

\[
\begin{bmatrix}
-1 + 3x^2y & x^3 \\
-2x & 1
\end{bmatrix}
\]

This implies that the \((\text{trace}(DF), \det(DF))\) for the three equilibrium points is \((0, -1), (3, 4), (3, 4)\). I.e., the equilibrium points of the system are a saddle (at \((0, 0)\)) and two spiral sources (at \((\pm 1, 1)\)).

2. Consider the equation:

\[
\begin{bmatrix}
\dot{x} \\
\dot{y}
\end{bmatrix} = \begin{bmatrix}
-x^3 - y^5 \\
x - y^5
\end{bmatrix}
\]

Construct a Lyapunov function at \((0, 0)\), and prove that \((0, 0)\) is an asymptotically stable equilibrium.

Solution: The Ansatz \( L = x^{2m} + by^{2n} \), yields \( \dot{L} = -2mx^{2m+2} - 2m^{2m-1}y^5 + 2bnxy^{2n-1} - 2nym^{2n+4} \). We see that if \( 2m - 1 = 1, 2n - 1 = 5 \) and \( b = m/n \), i.e., \( m = 1, n = 3, b = 1/3 \), then \( \dot{L} = -2x^4 - 2y^{10} < 0 \). I.e., \( L \) is a strict Lyapunov function, and hence \((0,0)\) is an asymptotically stable equilibrium point.

3. Solve the equation:

\[ u_x + x^2u_y = x \]

given the initial condition \( u(0, s) = s \). What is the domain of the solution?

Solution: The transversality condition is \( J = 1 \times 1 - s^2 \times 0 = 1 \neq 0 \), which is satisfied for all initial values. The characteristic equations are \( x_t = 1, y_t = x^2, u_t = x \). Solve the equation for \( x \) first: \( x_t = 1, x(0) = 0 \) implies \( x(t, s) = t \). So the other two equations are \( y_t = t^2, y(0) = s \) and \( u_t = t, u(0) = s \), which have the solutions \( y(t, s) = t^3/3 + s \) and \( u(t, s) = t^2/2 + s \). \( x = t \) implies \( s = y - x^3/3 \), which yields the solution \( u(x, y) = x^2/2 - x^3/3 + y \), which is defined for all \((x, y) \in \mathbb{R}^2\).
4. Solve the equation:

\[-uu_x + yu_y = x, \quad x > 0,\]
given the initial condition \(u(s, 1) = 0\). What is the domain of the solution?

**Solution:** The transversality condition is \(J = -0 * 0 - 1 * 1 = -1 \neq 0\), which is satisfied for all initial values. The characteristic equations are \(x_t = -u, y_t = y, u_t = x\).

We solve the \((x, u)\) system, which is a linear system with solutions \(x(t) = a \cos t - b \sin t, u(t) = a \sin t + b \cos t\), the initial values \(x(0) = s, u(0) = 0\) yield \(a = s, b = 0\).

Thus, \(x(t) = s \cos t, u(t) = s \sin t\). Note that the assumption \(x > 0\) implies that \(s > 0\)
(from the initial condition) and hence that \(-\pi/2 < t < \pi/2\) (from the equation for \(x\)). The \(y\) equation has the solution \(y(t) = e^t\), which means that \(e^{-\pi/2} < y < e^{\pi/2}\) for the solutions that we get from the characteristics. From \(y = e^t\) we get \(t = \ln y\). Since \(-\pi/2 < t < \pi/2\), we can solve for \(s\) from the equation for \(x\): \(s = x/ \cos t = x/ \cos(\ln y)\).

Finally, by inserting \(s\) and \(t\) into the equation for \(u\), we get:

\[u = x \tan(\ln y), \quad x > 0, \quad e^{-\pi/2} < y < e^{\pi/2}.\]