Geometric notions of space complexity for the word problem

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May 2006

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Filling length as \textit{SPACE}

\(\Gamma\) a group with finite presentation \(\langle A \mid R \rangle\)

\(w\) a word representing 1

\(\text{FL}(w)\) is the minimal \(L\) such that \(w\) can be converted to the empty word \(\epsilon\) through words of length at most \(L\) by

– applying relators
– freely reducing
– freely expanding.

Filling length function \(\text{FL} : \mathbb{N} \to \mathbb{N}\)

\(\text{FL}(n) = \max \{\text{FL}(w) \mid w = 1 \text{ in } \Gamma \text{ and } \ell(w) \leq n\} \)
Example

\[
\langle a, b \mid a^{-1}b^{-1}ab \rangle
\]

\[
\begin{align*}
& baba^{-2}bab^{-3} \\
\quad & \downarrow \\
& baba^{-2}abb^{-3} \\
\quad & \downarrow \\
& baba^{-1}b^{-1}b^{-1} \\
\quad & \downarrow \\
& bb^{-1} \\
\quad & \downarrow \\
& \varepsilon
\end{align*}
\]

\[\text{FL}(n) \simeq n\]
Filling length via geometry

For a loop $\rho$ in a simply connected metric space $X$,

$$\text{FL}(\rho) = \inf \left\{ L \left| \exists \text{ a based null–homotopy of } \rho \text{ through loops of length } \leq L \right. \right\}$$

$$\text{FL}(\ell) = \sup \{ \text{FL}(\rho) | \text{ loops } \rho \text{ of length at most } \ell \}$$
The **Cayley 2-complex** of

\[ \langle a_1, \ldots, a_m \mid r_1, \ldots, r_n \rangle \]

is the universal cover of
Example

\[ \langle a, b \mid [a, b] \rangle = \mathbb{Z}^2 \]
Example

\[ \langle a, b \mid b^{-1}ab = a^2 \rangle \]
\[ \mathbb{Z}^3 = \langle a, b, c \mid [a, b], [b, c], [c, a] \rangle \]
Combinatorial null-homotopy moves

1-cell collapse

2-cell collapse

1-cell expansion

fragmentation
Does allowing free null-homotopies change filling length?

I.e. allowing cyclic conjugation

$\text{FFL}(\rho) = \text{Free filling length}$
Theorem. There is a finitely presented group $\mathcal{P}$ with a family of words $w_n$ representing $1$, such that

\[
\ell(w_n) \simeq n \\
\text{FFL}(w_n) \simeq n \\
\text{FL}(w_n) \simeq 2^n.
\]

Theorem. There is a closed Riemannian manifold with a family of null-homotopic loops $\rho_n$ such that

\[
\ell(\rho_n) \simeq n \\
\text{FFL}(\rho_n) \simeq n \\
\text{FL}(\rho_n) \simeq 2^n.
\]
Generators: \( a, b, r, s, t \)

Relations: \( b^{-1}aba^{-2}, [t, a], [r, at], [r, s], [s, t] \)

\[
w_n := [s, (b^{-n}a^{-1}b^n)r(b^{-n}ab^n)]
\]
**Theorem.** The filling functions

\[ FL, FFL, FFFL : \mathbb{N} \to \mathbb{N} \]

for \( \mathcal{P} \) satisfy

\[ FL(n) \simeq FFL(n) \simeq 2^n \]
\[ FFFL(n) \simeq n. \]
FFFL = Free and fragmenting filling length
Open problem.
Does there exist a finite presentation for which $\text{FL}(n) \not\cong \text{FFL}(n)$?