Genus minimizing knots in rational homology spheres

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“What’s Next?”
The mathematical legacy of Bill Thurston
Cornell University,
June 23–27, 2014
- Thurston norm
- Heegaard Floer homology
- The rational genus bound
- $\mathbb{Z}_2$–Thurston norm and triangulations
Thurston norm

Heegaard Floer homology

The rational genus bound

$\mathbb{Z}_2$–Thurston norm and triangulations
The (Seifert) genus of a knot $K \subset S^3$ is defined to be

$$g(K) = \min \{ g(F) \mid F \text{ is a Seifert surface for } K \}.$$
Genus bounds from the Alexander polynomial

Let

\[ \Delta_K(t) = a_0 + \sum_{i=1}^{n} a_i(t^i + t^{-i}) \]

be the symmetrized Alexander polynomial of a knot $K$, where $a_n \neq 0$. 

Proposition

The genus of $K$ is bounded below by the degree of $\Delta_K$, namely

\[ \deg \Delta_K := n \leq g(K) \]

This bound is not always sharp. In fact, there are infinitely many nontrivial knots with $\Delta_K \equiv 1$. 

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Thurston Norm (Thurston, 1976)

Let $S$ be a compact oriented surface with connected components

$$S_1, \ldots, S_n.$$ 

We define

$$\chi^-(S) = \sum \max\{0, -\chi(S_i)\}.$$ 

Let $M$ be a compact oriented 3–manifold, $A$ be a homology class in $H_2(M; \mathbb{Z})$ or $H_2(M, \partial M; \mathbb{Z})$. The Thurston norm $x(A)$ of $A$ is defined to be the minimal value of $\chi^-(S)$, where $S$ runs over all the properly embedded oriented surfaces in $M$ with $[S] = A$.

Any Seifert surface can be regarded as a properly embedded surface in $M = S^3 \setminus \text{int}(\nu(K))$, where $\nu(K)$ is a tubular neighborhood of $K$ in $S^3$. Let $A$ be a generator of $H_2(M, \partial M)$ $\approx \mathbb{Z}$, then

$$x(A) = \begin{cases} 
0, & \text{when } K \text{ is the unknot,} \\
2g(K) - 1, & \text{otherwise.} 
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A semi-norm

The function $x$ has the following basic properties:

- (Homogeneity) $x(nA) = |n| \cdot x(A)$, $n \in \mathbb{Z}$.
- (Triangle Inequality) $x(A + B) \leq x(A) + x(B)$. 

Thus one can extend $x$ homogenously and continuously to a semi-norm $x$ on $H^2(M; \mathbb{R})$ or $H^2(M, \partial M; \mathbb{R})$.

It is only a semi-norm because $x$ vanishes (exactly) on the subspace of $H^2$ generated by the homology classes of spheres, disks, tori and annuli.

McMullen: there is a lower bound to $x$ in terms of the Alexander polynomial of $M$.

The unit ball of $x$ is a convex polytope which is symmetric in the origin, also called the Thurston polytope.
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A page from Thurston’s paper “A norm for the homology of 3–manifolds”

Figure 1
Lines of the form $nx + my = 1/2$ where $n$ and $m$ are integers.
Any convex polygon in this network which is symmetric in the origin is the unit sphere in $\mathbb{H}_2(M)$, for some 3–manifold $M$.

This computer drawn picture was prepared by Nathaniel Thurston.
Theorem (Thurston)
Suppose that $M$ is a compact oriented 3–manifold. Let $\mathcal{F}$ be a taut foliation over $M$ such that each component of $\partial M$ is either a leaf of $\mathcal{F}$ or transverse to $\mathcal{F}$, and in the latter case $\mathcal{F}|\partial M$ is also taut. Then every compact leaf of $\mathcal{F}$ attains the minimal $\chi$–in its homology class.

The proof uses a technique independently developed by Roussarie and Thurston (in his thesis).

Gabai proved a converse to the above theorem.

Theorem (Gabai)
Suppose that $M$ is a compact oriented irreducible 3–manifold with (possibly empty) boundary consisting of tori. Let $S \subset M$ be a properly embedded surface which minimizes $\chi$–in the homology class of $[S] \in H_2(M,\partial M)$. Then there exists a taut foliation $\mathcal{F}$ over $M$ such that $S$ consists of compact leaves of $\mathcal{F}$. 
Thurston norm and taut foliations

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\section*{Spin$^c$ structures}

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$$c_1(s_1) - c_1(s_2) = 2(s_1 - s_2).$$
Heegaard Floer homology

Let $Y$ be a closed, oriented, connected 3-manifold, $s \in \text{Spin}^c(Y)$. Ozsváth and Szabó defined a package of invariants associated with $(Y, s)$: $\widehat{HF}(Y, s), HF^+(Y, s)$.
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For each $Y$, there are only finitely many $s \in \text{Spin}^c(Y)$ such that $\widehat{HF}(Y, s) \neq 0$ (iff $HF^+(Y, s) \neq 0$).
Heegaard Floer homology detects the Thurston norm

Theorem (Ozsváth–Szabó)

Suppose that $Y$ is a closed oriented 3–manifold, $A \in H_2(Y)$. Then

$$x(A) = \max \left\{ \langle c_1(s), A \rangle \mid s \in \text{Spin}^c(Y), \; HF^+(Y, s) \neq 0 \right\}.$$
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This theorem can be viewed as a generalization of McMullen’s Alexander bound of the Thurston norm.
Knot Floer homology and Seifert genus

There are also versions of the previous theorem for manifold with torus boundary.

When $K$ is a knot in $S^3$, its knot Floer homology is a finitely generated bigraded abelian group

$$\widehat{HFK}(K) = \bigoplus_{i,j} \widehat{HFK}_j(K, i).$$

Here $i$ is called the “Alexander grading”, and $j$ is the “Maslov grading” or “homological grading”. This invariant was introduced by Ozsváth–Szabó and Rasmussen.
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This theorem has been generalized to links in $S^3$ (Ozsváth–Szabó) and in arbitrary closed 3–manifold (Ni).
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Later developments allow us to bypass these contact and symplectic results (Juhász, Kronheimer–Mrowka, Ni).
▶ Thurston norm
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▶ The rational genus bound
▶ $\mathbb{Z}_2$–Thurston norm and triangulations
Let $K \subset Y$ be a rationally null-homologous knot, namely, $[K] = 0 \in H_1(Y; \mathbb{Q})$. 

Rational Seifert surface
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A properly embedded oriented surface $F \subset M = Y \setminus \text{int}(\nu(K))$ is called a rational Seifert surface for $K$, if $\partial F$ consists of coherently oriented parallel curves on $\partial M$, and the orientation of $\partial F$ is coherent with the orientation of $K$. 
Rational genus

Calegari–Gordon: The rational genus of $K$ is defined to be

$$g_r(K) = \min_F \frac{\chi-(F)}{2|[\mu] \cdot [\partial F]|},$$

where $F$ runs over all the rational Seifert surfaces for $K$, and $\mu \subset \partial \nu(K)$ is the meridian of $K$. 
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When $K$ is null-homologous and nontrivial,

$$g_r(K) = \frac{2g(K) - 1}{2} = g(K) - \frac{1}{2}.$$
A function on $\text{Tors}H_1(Y)$

Given a torsion homology class $a \in \text{Tors}H_1(Y)$, let

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Turaev gave a lower bound to $\Theta(a)$ in terms of his torsion function. He asked whether this lower bound is sharp for lens spaces.
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Correction terms

For a rational homology sphere $Y$, there is an absolute Maslov $\mathbb{Q}$–grading on $HF^+(Y, s)$. This $d(Y, s) \in \mathbb{Q}$ is called the correction term of $(Y, s)$. 
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For a rational homology sphere $Y$, there is an absolute Maslov $\mathbb{Q}$–grading on $HF^+(Y, s)$. In this case, there is a canonical subgroup in $HF^+(Y, s)$ which is isomorphic to $H_{* - d}(\mathbb{C}P^\infty)$ for some $d = d(Y, s) \in \mathbb{Q}$. 
Corruption terms

For a rational homology sphere $Y$, there is an absolute Maslov $\mathbb{Q}$–grading on $HF^+(Y, \xi)$.

In this case, there is a canonical subgroup in $HF^+(Y, \xi)$ which is isomorphic to $H_{*-d}(\mathbb{C}P^\infty)$ for some $d = d(Y, \xi) \in \mathbb{Q}$. This $d(Y, \xi) \in \mathbb{Q}$ is called the correction term of $(Y, \xi)$. 
Lens spaces

Let the lens space $L(p, q)$ be oriented as the $\frac{p}{q}$-surgery on $S^3$. The correction terms of $L(p, q)$ can be computed by the recursive formula:

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\begin{align*}
d(S^3, 0) &= 0, \\
d(L(p, q), i) &= -\frac{1}{4} + \left(2i + 1 - p - q\right)\frac{2}{4pq} - d(L(q, r), j),
\end{align*}
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where $0 \leq i < p$, $r$ and $j$ are the reductions modulo $p$ of $q$ and $i$, respectively. There are also closed formulas for $d(L(p, q), i)$ involving Dedekind sums (Némethi, Tange) or Dedekind–Rademacher sums (Jabuka–Robins–Wang).
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The rational genus bound

Theorem (Ni–Wu)

Suppose that $Y$ is a rational homology 3–sphere, $K \subset Y$ is a knot, $F$ is a rational Seifert surface for $K$. Then

$$1 + \frac{-\chi(F)}{|[\partial F] \cdot [\mu]|} \geq \max_{s \in \text{Spin}^c(Y)} \left\{ d(Y, s + \text{PD}[K]) - d(Y, s) \right\}.$$

The right hand side of the inequality only depends on the manifold $Y$ and the homology class of $K$, so it gives a lower bound for $1 + \Theta(a)$ for the homology class $a = [K]$. 
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Suppose that $Y$ is a rational homology 3–sphere, $K \subset Y$ is a knot, $F$ is a rational Seifert surface for $K$. Then

$$1 + \frac{-\chi(F)}{||[\partial F] \cdot [\mu]||} \geq \max_{s \in \text{Spin}^c(Y)} \{d(Y, s + \text{PD}[K]) - d(Y, s)\}.$$ 

The right hand side of the inequality only depends on the manifold $Y$ and the homology class of $K$, so it gives a lower bound for $1 + \Theta(a)$ for the homology class $a = [K]$. 
Floer simple knots in L-spaces

A rational homology sphere $Y$ is an L-space if

$$\text{rank } \widehat{HF}(Y) = |H_1(Y; \mathbb{Z})|.$$
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Given a 3–manifold $Z$, a rationally null-homologous knot $K \subset Z$ is a Floer simple knot if

$$\text{rank } \widehat{HFK}(Z,K) = \text{rank } \widehat{HF}(Z),$$

where $\widehat{HFK}(Z,K)$ is the knot Floer homology of $K$. 

Corollary (Ni–Wu)

The bound for $\Theta$ via correction terms is sharp for the homology classes represented by Floer simple knots in L-spaces. In fact, Floer simple knots in L-spaces attain the minimal values of the rational genus in their homology classes.
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Simple knots

Let $U_1 \cup U_2$ be the genus one Heegaard splitting of $L(p, q)$. Let $D_i$ be the meridian disk of $U_i$, then $\partial D_1 \cap \partial D_2$ consists of $p$ points.
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Pick any two points in $\partial D_1 \cap \partial D_2$, connecting them with arcs $\gamma_1 \subset D_1$ and $\gamma_2 \subset D_2$. The knot $\gamma_1 \cup \gamma_2$ is called a simple knot in $L(p, q)$.
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Computing $\Theta$ for lens spaces

Simple knots in lens spaces are Floer simple.
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Simple knots in lens spaces are Floer simple. Thus the $\Theta$ of lens spaces can be computed from the correction terms, and simple knots are genus minimizers in their homology classes.

For example, in $L(p,1)$, for the homology class $a \in \{0, 1, \ldots, p-1\}$, $\Theta(a) = \max\{0, a(p-a)\}$.
Computing $\Theta$ for lens spaces

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Our computation shows that $\Theta$ can be quite large for lens spaces. For example, in $L(p, 1)$, for the homology class $a \in \{0, 1, \ldots, p - 1\}$,

$$\Theta(a) = \max\{0, \frac{a(p - a)}{p} - 1\}.$$

So if $a \sim \frac{p}{2}$, $\Theta(a) \sim \frac{p}{4}$. 
Theorem (Hedden, Rasmussen)

Suppose that $L(p, q)$ is obtained by $p$-surgery on a knot $K \subset S^3$, then the dual knot $K' \subset L(p, q)$ is a Floer simple knot, and it is a rational genus minimizer in its homology class.
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There are similar results for lens space surgery on knots in lens spaces (studied by Boileau–Boyer–Cebanu–Walsh) or $S^1 \times S^2$ (studied by Cebanu, Baker–Buck–Lecuona).

Thus it is an interesting problem to find all the rational genus minimizers in lens spaces.
Uniqueness of genus minimizers

When $\Theta(a) < \frac{1}{2}$ and the minimal genus rational Seifert surface has only one boundary component, Baker proved that any rational genus minimizer in the homology class $a$ must have bridge number 1.
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Rasmussen asked the question whether simple knots are the unique rational genus minimizers in lens spaces.
Non-uniqueness of genus minimizers

Theorem (Greene–Ni)

There are infinitely many triples \((p, q, a)\), such that there are non-simple rational genus minimizers in the homology class \(a \in H_1(L(p, q))\). Moreover, there exist infinitely many triples \((p, q, a)\), such that there are infinitely many rational genus minimizers in the homology class \(a \in H_1(L(p, q))\).
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All the examples we have found have large \(\Theta\). It is possible that the uniqueness holds when \(\Theta\) is small. For example, when \(\Theta < \frac{1}{2}\) or even \(\Theta < 1\).
The simplest example we have found is the \((1, 2)\)–cable of the \((1, 2)\)–torus knot in \(L(8, 1)\). The simple knot in this homology class is the \((1, 4)\)–torus knot.
Thurston norm

Heegaard Floer homology

The rational genus bound

$\mathbb{Z}_2$–Thurston norm and triangulations
Non-orientable genus

**Fact:** Any non-orientable surface $\Pi \subset Y$ represents a nonzero class in $H_2(Y; \mathbb{Z}_2)$. Conversely, any nonzero class in $H_2(Y; \mathbb{Z}_2)$ is represented by a non-orientable surface.
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This $h(Y, A)$ is closely related to the the so-called $\mathbb{Z}_2$-Thurston norm $\|A\|_{\mathbb{Z}_2}$ of $A$. Similar to the Thurston norm, $\|A\|_{\mathbb{Z}_2}$ is defined to be the minimal $\chi_-$ of (not necessarily orientable) surfaces representing $A$. 
Non-orientable genus and $\Theta$

When the order of $[K] \in H_1(Y; \mathbb{Z})$ is 2, any rational Seifert surface $F$ gives rise to a closed non-orientable surface $\hat{F} \subset Y$, such that $\beta([\hat{F}]) = [K]$, where

$$\beta : H_2(Y; \mathbb{Z}_2) \to H_1(Y; \mathbb{Z})$$

is the Bockstein homomorphism. This relates $\Theta([K])$ with the non-orientable genus of $\hat{F}$. 
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**Proposition**

*Let $Y$ be a rational homology 3–sphere. Given a nonzero class $A \in H_2(Y; \mathbb{Z}_2)$, if $h(Y, A) \geq 2$, then we have*

$$h(Y, A) = 2\Theta(\beta(A)) + 2.$$
Corollary

Let $Y$ be a rational homology 3–sphere, $A \in H_2(Y; \mathbb{Z}_2)$, then

$$h(Y, A) \geq 2 \max_{s \in \text{Spin}^c(Y)} \left\{ d(Y, s + \text{PD} \circ \beta(A)) - d(Y, s) \right\}.$$
Corollary

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Levine–Ruberman–Strle proved that the bound in the above corollary is also a lower bound to the non-orientable genus in $Y \times I$. 
More computations

Ni–Wu:
Let $L$ be the closure of the pure 3–braid

$$\sigma = \sigma_1 \sigma_2^{-2a_1} \sigma_1 \sigma_2^{-2a_2} \cdots \sigma_1 \sigma_2^{-2a_{2n-1}} \sigma_1 \sigma_2^{-2a_{2n}},$$

where $a_i, n > 0$, and $\Sigma(L)$ be the double branched cover of $S^3$ branched along $L$. Then the $\mathbb{Z}_2$–Thurston norms of the three nonzero homology classes in $H_2(\Sigma(L); \mathbb{Z}_2)$ are

$$\sum_{i \text{ odd}} a_i + n - 2,$$
$$\sum_{i \text{ even}} a_i + n - 2,$$
$$2n \sum_{i=1} a_i - 2.$$
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**Theorem (Jaco–Rubinstein–Tillmann)**

Let $Y$ be a closed, orientable, irreducible, atoroidal, connected 3–manifold with triangulation $\mathcal{T}$. Let $H \subset H_2(Y; \mathbb{Z}_2)$ be a rank 2 subgroup, then

$$|\mathcal{T}| \geq 2 + \sum_{A \in H} ||A||_{\mathbb{Z}_2}.$$
The theorem of Ni–Wu implies that $\| \cdot \|_{\mathbb{Z}_2}$ is bounded below in terms of correction terms. As a result, $C(Y)$ is bounded below in terms of correction terms in the cases discussed in Jaco–Rubinstein–Tillmann.
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In the previous example of $\Sigma(L)$, $H_1(\Sigma(L); \mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$, and

$$
C(\Sigma(L)) \geq 2 \sum_{i=1}^{2n} a_i + 2n - 4.
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On the other hand, we can construct a triangulation of $\Sigma(L)$ with

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Thank you!