Cyclic branched coverings of knots and a characterization of S^3

The mathematical legacy of Bill Thurston, June 24, 2014

Joint with Clara Franchi, Mattia Mecchia, Luisa Paoluzzi & Bruno Zimmermann

Orbifolds

Orbifolds are natural generalizations of manifolds, and can be roughly described as spaces which locally look like quotients of manifolds by finite group actions.

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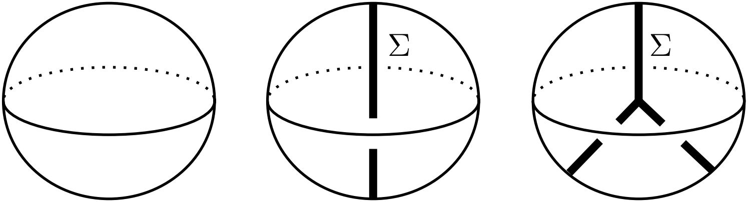
Orbifold Theorem

In dimension 3, an orbifold is a metrizable space in which each point has a neighbourhood modelled on a quotients of the ball B^3 by a finite subgroup of $SO(3)$.

The set of points having non-trivial *local isotropy group* is called the *singular locus* of the orbifold. It is a trivalent graph.



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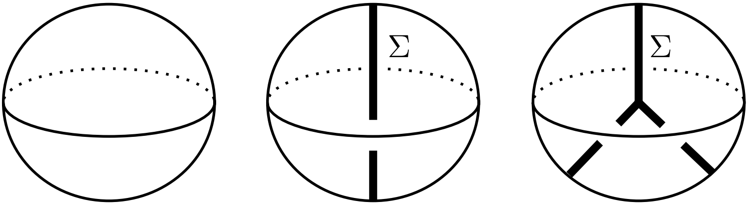

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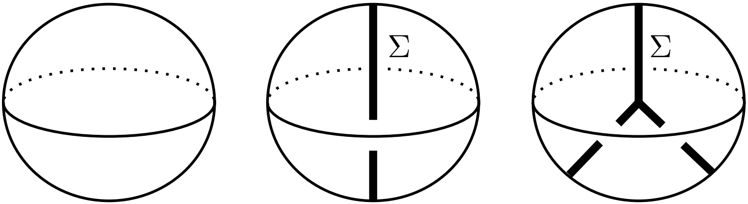

Cyclic branched coverings

A classical way to construct closed 3-manifolds is by taking finite cyclic coverings of the 3-sphere S^3 branched along knots.

The n-fold cyclic covering $M_n(K)$ of S^3 branched along K admits a periodic diffeomorphism ϕ of order n corresponding to the covering translation.

The quotient $M_n(K)/ < \phi >$ is an orbifold $\mathcal{O}(K, n)$ with underlying space S^3, singular locus K and local model for all singular points a *football*.

The projection $M_n(K) \to \mathcal{O}(K, n)$ corresponds to the orbifold n-fold cyclic covering of $\mathcal{O}(K, n)$

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Orbifold Theorem

Thm (W. Thurston's Orbifold Theorem)

A compact orientable 3-orbifold without bad 2-suborbifold has a canonical geometric decomposition along a finite collection of spherical and euclidean essential 2-suborbifolds.

Corollary

Let $K \subset S^3$ be a knot :
(1) *$M_n(K)$ has a canonical decomposition into geometric pieces on which the covering translation group acts equivariantly by isometries.*

(2) *If $S^3 \setminus K$ admits a completre hyperbolic structure, then for $n \geq 3$ $M_n(K)$ admits a hyperbolic structure, except when $n = 3$ and K is the figure-8 knot where it is Euclidean.*

(3) *(Smith conjecture) K is the unknot iff $M_n(K) \cong S^3$ for some $n \geq 2$.*

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Cyclic branched coverings

Given $M = M_n(K)$ a prime manifold there are some strong relationship between M, K and n.

Thm (A. Salgueiro)

M and K determine n when n is prime.

Thm (B-Paoluzzi ; Zimmermann)

Given M and n an odd prime number, there are at most two knots K and K' such that $M \cong M_n(K) \cong M_n(K')$.

Moreover this can occure for at most two odd prime numbers n.

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A well-known property of the standard sphere S^3 is to admit hyperelliptic rotations of any order.

Due to W. Thurston's orbifold theorem, one has :

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Given a closed orientable 3-manifold M :
(1) *There are only finitely many knots $K \subset S^3$ such that $M \cong M_n(K)$ for some $n \geq 2$.*

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2-fold coverings

Remark

A priori, the number of knots in S^3 having M as a cyclic branched covering can be arbitrarily large.

For example when M is not prime or, when $n = 2$ and M is not hyperbolic.

For a hyperbolic manifold Marco Reni proved :

Thm (M. Reni)

A closed orientable hyperbolic 3-manifold. M is a 2-fold covering of S^3 branched along a knot for at most 9 distinct knots.

This bound is sharp (K. Kawauchi)

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Odd prime orders

Thm (BFMPZ)

The group $Diff^+(M)$ of orientation preserving diffeomorphisms of a closed, orientable, connected, irreducible 3-manifold $M \ncong S^3$ contains at most 6 conjugacy classes of cyclic subgroups generated by a hyperelliptic rotation of odd prime order.

A straightforward corollary is :

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A closed orientable connected irreducible 3-manifold. M is a cyclic covering of S^3 with prime odd order and branching set a knot for at most 6 distinct knots.

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Characterization of S^3

The decomposition of a closed manifold as a connected sum of prime manifolds and the equivariant sphere theorem implies :

Corollary

A closed connected orientable 3-manifold M is homomorphic to S^3 iff it admits 7 hyperelliptic rotations with distinct odd prime orders.

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The requirement that the rotations are hyperelliptic is essential since the Brieskorn homology sphere $\Sigma(p_1, \ldots, p_n)$, $n \geq 4$, admits n rotations of pairwise distinct odd prime orders but with non-trivial quotient.

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Thurston orbifold theorem and some surgery arguments allow to reduce the proof to the case of a finite group of diffeomorphisms acting on M :

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In a closed hyperbolic manifold each hyperelliptic rotation is conjugated into the group of isometries which is finite. Combining Marco Reni's and our results :

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Sylow subgroups

One interesting aspect of the proof of this result is the use of finite group theory and of the classification of finite simple groups.

The proof splits in various cases, according to the structure of the normalizer of the p-Sylow subgroups, containing a hyperelliptic rotation of odd prime order p.

This structure is reflected in the symmetries of the orbifold $\mathcal{O}_n(K)$.

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If $G \subset Diff^+(M)$ is a finite group, one can choose a Riemannian metric on M which is invariant by G.

The normaliser $\mathcal{N}_G(<\phi>)$ of a (hyperelliptic) rotation ϕ in G must leave the circle of fixed points $Fix(\phi)$ invariant.

Hence $\mathcal{N}_G(<\phi>)$ is a finite subgroup of $\mathbb{Z}/2 \ltimes (\mathbb{Z}_a \oplus \mathbb{Z}_b)$, for some non negative integer a and b :

The element of order 2 acts by sending each element of the product $\mathbb{Z}_a \oplus \mathbb{Z}_b$ to its inverse.

The elements of $\mathcal{N}_G(<\phi>)$ are precisely those that rotate about $Fix(\phi)$, translate along $Fix(\phi)$, or inverse the orientation of $Fix(\phi)$.

In the last case the elements have order 2 and non empty fixed-point set meeting $Fix(\phi)$ in two points.

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Lemma

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(1) *The Sylow p-subgroup S_p of G is either cyclic or of the form $\mathbb{Z}/p^\alpha \oplus \mathbb{Z}/p^\beta$.*

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Steps of the proof

First step : Prove the result for $G \subset \mathit{Diff}^+(M)$ a solvable finite group. The bound in this case is 3

Second step : study solvable normal covers of the finite group G.

Let G be a non-solvable finite group and π the set of odd primes dividing $|G|$. A collection $\mathcal{C}$ of solvable subgroups of G is a *solvable normal* π*-cover* of G if every element of G of prime order belongs to $\cup_{H \in \mathcal{C}}$ and for every $g \in G, H \in \mathcal{C}$ $gHg^{-1} \in \mathcal{C}$.

We denote by $\gamma_\pi^s(G)$ the smallest number of conjugacy classes of subgroups in a solvable normal π-cover of G.

Since Sylow subgroups are solvable, $\gamma_\pi^s(G) \leq |\pi|$.

For q an odd prime power, $\gamma_\pi^s(PSL_2(q)) = 2$.

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For q an odd prime power, $\gamma^s_\pi(PSL_2(q)) = 2$.

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Second step : study solvable normal covers of the finite group G.

Let G be a non-solvable finite group and π the set of odd primes dividing $|G|$. A collection $\mathcal{C}$ of solvable subgroups of G is a *solvable normal π-cover* of G if every element of G of prime order belongs to $\cup_{H \in \mathcal{C}}$ and for every $g \in G, H \in \mathcal{C}$ $gHg^{-1} \in \mathcal{C}$.

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Solvable case

Proposition

Let $G \subset Diff^+(M)$ be a finite solvable group acting on a 3-manifold $M \neq S^3$. Then :

(1) *If G contains $n \geq 3$ hyperelliptic rotations of odd prime orders, then, up to conjugacy, they commute.*

(2) *Up to conjugacy, G contains at most three hyperelliptic rotations of odd prime orders.*

(3) *Either their orders are pairwise distinct or there are at most two such conjugacy classes of rotations.*

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If M admits four commuting hyperelliptic rotations with pairwise distinct odd prime orders.

Fix one of these rotations ϕ and consider the covering projection $\pi : M \longrightarrow \mathcal{O}_p(K)$ branched along the knot $K = \pi(Fix(\phi))$.

The three remaining rotations commute with ψ and thus induce 3 **full rotational symmetries** of K (i.e. with quotient *a trivial knot*) and distinct prime orders.

Thm (B-Paoluzzi)

A knot K which admits three full rotational symmetries with pairwise distinct orders > 2, is the unknot.

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$\mathbb{Z}$-Homology spheres

Corollary

A finite subgroup $G \subset Diff^+(M)$ of a $\mathbb{Z}HS$ $M \ncong S^3$ contains at most 3 conjugacy classes of cyclic subgroups generated by a hyperelliptic rotation of prime odd order.

The number 3 is realized by a Briekorn sphere $\Sigma(p,q,r) = \{X^p + Y^q + Z^q = 0\} \cap \{|X|^2 + |Y|^2 + |Z|^2 = 1\}$ where p, q, r are 3 distinct odd primes.

It is also realized by some hyperbolic $\mathbb{Z}$HS.

3 is expected to be the maximal number in any cases.

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In the $\mathbb{Z}$HS case, the proof uses strongly the restrictions on finite groups acting on integral homology 3-spheres.

Lemma

Let M be a $\mathbb{Z}$HS. If a finite subgroup $G \subset Diff^+(M)$ contains a rotation of prime order $p \geq 7$, then G is solvable.

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According to Mecchia and Zimmermann a finite group G acting on a $\mathbb{Z}$HS is solvable or isomorphic to a group of the following list :

$\mathbb{A}_5$, $\mathbb{A}_5 \times \mathbb{Z}/2$, $\mathbb{A}_5^* \times_{\mathbb{Z}/2} \mathbb{A}_5^*$ or $\mathbb{A}_5^* \times_{\mathbb{Z}/2} C$.

- $\mathbb{A}_5$ is the dodecahedral group (alternating group on 5 elements), $\mathbb{A}_5^*$ is the binary dodecahedral group (isomorphic to $SL_2(5)$).

- C is a solvable group with a unique involution and which acts freely on M.

- $\times_{\mathbb{Z}/2}$ denotes a central product, i.e. the quotient of the two factors in which the two central involutions are identified.

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Since the elements of C act freely, they cannot be rotations.

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General case

A case by case analysis using the structure of the maximal semisimple normal subgroup $E(G)$ of G shows that :

Either there are at most 6 conjugacy classes of hyperelliptic involution

or $\gamma_{\pi}^{s}(G) \leq 6$.

Moreover when $\gamma_{\pi}^{s}(G) > 2$, each solvable subgroup of the normal cover of G contains at most one conjugacy class of hyperelliptic rotation.

Semisimple means perfect and the factor group $E(G)/Z(E(G)$ is a product of non abelian simple groups. That is where the classification of simple groups occurs.

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Questions

1- Is 3 the sharp bound for the number of conjugacy classes of hyperelliptic rotations with odd prime orders ?

2- For hyperbolic manifolds is there a uniform bound on the number of conjugacy classes of hyperelliptic rotations without any assumption on their orders ?

3- What about commensurability classes of the orbifolds $\mathcal{O}_n(K)$?

Are there finitely many such orbifolds in the same commensurability class ?

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