Math6720s11 - Homework 3

Due: 7 March 2011.

Please make all do effort to type your solutions to this problem set. This is so that you will be able to cleanly fix any mistakes that you find prior to submission and so that the grader can easily read what you have written. Thank you.

Exercise 1. Exercise 8.5.4 Durrett. If $T = \inf\{t : B_t \notin (a, b)\}$ where $a < 0 < b$ and $a \neq -b$, then $T$ and $B_T^2$ are not independent, so the calculation of $ET^2$ from Theorem 8.5.9 does not work. Instead use the Cauchy-Schwarz inequality to estimate $E(TB_T^2)$ and conclude $ET^2 \leq CEB_T^2$ independently of $a, b$.

Exercise 2. Exercise 8.6.1 Durrett. Use exercise 8.5.4 to conclude that $E(T_{U,V}^2) \leq CEX^4$.

Exercise 3. Exercise 8.5.5 Durrett. Find a martingale of the form $B_t^6 - c_1 t B_t^4 + c_2 t^2 B_t^2 - c_3 t^3$ and use it to compute $ET^3$. Where $T = \inf\{t : B_t \notin (-a,a)\}$

Exercise 4. Exercise 8.5.6 Durrett. Show that $(1+t)^{-1/2} e^{B_t^2/(1+t)}$ is a martingale and use this to conclude that

$$
\limsup_{t \to \infty} \frac{B_t}{(1+t) \log(1+t)^{1/2}} \leq \frac{1}{\sqrt{2}} \quad \text{a.s.}
$$

Exercise 5. Given a centered\footnote{mean zero} random variable $X$, show that there exists centered random variables $X_n$ taking only finitely many values, such that $X_n$ converge to $X$ in law and for $\Psi_n(x) = E(X_n|X_n \geq x)$ the embedding stopping times

$$
\tau_n = \inf\{t \geq 0 : \sup_{s \leq t} B_s \geq \Psi(B_t)\}
$$

converge almost surely to $\tau$. Conclude that $B_\tau$ has the same law as $X$, and $E\tau = EX^2$.

Note that this is a middle step in the proof of the Skorokhod representation theorem as we have proven it.