Math6710d11 - Homework 1

Due: 9 September 2011

Homework is preferred typed, but legibly handwritten is accepted. The TA is allowed to mark problems with “take read, can’t grade.” Late homework is not accepted without prior consent.

Exercise 1. (Durrett 1.1) Recall the definition of $S_d$ from Example 1.1.3. Show that $\sigma(S_d) = B(\mathbb{R}^d)$.

Exercise 2. (Durrett-ish 1.1)

1. Show that if $F_i \subset F_{i+1}$ are algebras that $\bigcup_{i=0}^{\infty} F_i$ is an algebra.
2. Show by counterexample that if $F_i \subset F_{i+1}$ are $\sigma-$algebras that $\bigcup_{i=0}^{\infty} F_i$ is not necessarily a $\sigma-$algebra.

Exercise 3. Let $([0,1], \mathcal{L}([0,1]), \lambda)$ be Lebesgue measure on $[0,1]$. Consider a second probability triple $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$ such that $\Omega_2 = \{1,2\}$, $\mathcal{F}_2$ consists of all subsets of $\Omega_2$, and $\mathbb{P}_2(\{1\}) = \frac{1}{2}$ and extended by additivity. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the product probability triple.

1. Express each of $\Omega$, $\mathcal{F}$, and $\mathbb{P}$ as explicitly as possible.
2. Find a set $A \in \mathcal{F}$ such that $\mathbb{P}(A) = \frac{1}{4}$.

Exercise 4. (Durrett 1.2.4) Show that if $F(x) = \mathbb{P}(X \leq x)$ is continuous then $Y = F(X)$ has a uniform distribution on $(0,1)$, that is if $y \in (0,1]$, $\mathbb{P}(Y \leq y) = y$.

Exercise 5. (Durrett 1.2.7) (i) Suppose $X$ has a density function $f$. Compute the distribution function of $X^2$ and then differentiate to find its density function. (ii) Work out the answer when $X$ has a standard normal distribution.

Exercise 6. (Durrett 1.5.5) If $g_m \geq 0$ then $\int \sum_m g_m d\mu = \sum_m \int g_m d\mu$.

Exercise 7. (Durrett 1.6.7) Let $\Omega = (0,1)$ equipped with Borel sets and Lebesgue measure. Let $\alpha \in (1,2)$ and $X_n = n^{\frac{1}{\alpha}} \chi_{(1/(n+1), 1/n]}$ which go to zero almost surely. Show that Theorem 1.6.8 can be applied with $H(x) = x$ and $g(x) = |x|^{2/\alpha}$, but that $X_n$ are not dominated by an integrable function (that is the conditions of DCT are not met).

Exercise 8. (Durrett 1.6.14) Let $X \geq 0$ but do not assume that $\mathbb{E}(1/X) < \infty$. Show that

$$\lim_{y \to \infty} y \mathbb{E}(1/X; X > y) = 0, \quad \lim_{y \downarrow 0} y \mathbb{E}(1/X; X > y) = 0.$$  

Exercise 9. (Durrett 1.7.3) Let $F, G$ be Stieltjes measure functions and let $\mu, \nu$ be the corresponding measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Show that

- $\int_{(a,b]} \{F(y) - F(a)\} dG(y) = (\mu \times \nu)(\{x, y\} : a < x \leq y \leq b)$.  
- $\int_{(a,b]} F(y) dG(y) + \int_{(a,b]} G(y) dF(y) = F(b)G(b) - F(a)G(a) + \sum_{x \in (a,b]} \mu(\{x\}) \nu(\{x\})$.  
- If $F = G$ is continuous then $\int_{(a,b]} 2F(y) dF(y) = F^2(b) - F^2(a)$.  

To see that the second term in 2 is needed, let $F(x) = G(x) = \mathbb{I}_{[0,\infty]}(x)$ and $a < 0 < b$.  

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