Logic and Computation in Finitely-Presentable Infinite Structures

Lecture 7: Automatic Structures

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Automatic Structures

Definitions
Worked examples
Basic Results
Recall

**Interpretation.** \( \mathcal{A} \) is FO-interpretable in \( \mathcal{B} \) means \( \mathcal{A} \) is isomorphic to some quotient structure

\[
(\Delta^\mathcal{B}; (\phi_i^\mathcal{B})_i)/\epsilon^\mathcal{B}.
\]

**Transfer of FO-decidability.** Then if the FO-theory of \( \mathcal{B} \) is decidable, so is the FO-theory of \( \mathcal{A} \).
Summary of definabilities

We have the following translations, implying the decidability of the corresponding theories.

\[
\begin{align*}
\text{WMSO}(\mathbb{N}, s) & \iff \text{automata on finite words} \iff \text{FO}(\mathcal{W}(2)) \\
\text{WMSO}(\{0, 1\}^*, s_0, s_1) & \iff \text{automata on finite trees} \iff \text{FO}(\mathcal{T}(2)) \\
\text{MSO}(\mathbb{N}, s) & \iff \text{automata on infinite words} \iff \text{FO}(\mathcal{W}^\omega(2)) \\
\text{MSO}(\{0, 1\}^*, s_0, s_1) & \iff \text{automata on infinite trees} \iff \text{FO}(\mathcal{T}^\omega(2))
\end{align*}
\]

For each notion of automaton \(\diamond \in \{\text{word, } \omega\text{-word, } \text{tree, } \omega\text{-tree}\}\), write \(\mathcal{U}_\diamond\) for the corresponding FO structure.
Automatic structures

Definition. A relational structure $\mathcal{A}$ is called $\Diamond$-automatic if it is FO-interpretable in $\mathcal{U}_\Diamond$.

Then, every $\Diamond$-automatic structure has decidable FO-theory.

Notation

- $W$-$\text{Aut}$: word-automatic structures
- $T$-$\text{Aut}$: tree-automatic structures
- $W^\omega$-$\text{Aut}$: $\omega$-word automatic structures
- $T^\omega$-$\text{Aut}$: $\omega$-tree automatic structures
Automatic presentation

**Definition.** If $\mathcal{A}$ is $\diamond$-automatic via

$$(\Delta^\mathcal{U}_\diamond; (\phi^\mathcal{U}_i)_i)/\epsilon^\mathcal{U}_\diamond,$$

then the structure

$$(\Delta^\mathcal{U}_\diamond; (\phi^\mathcal{U}_i)_i, \epsilon^\mathcal{U}_\diamond)$$

is an $\diamond$-automat**ic presentation** (of $\mathcal{A}$).

Note in particular, an automatic presentation is FO-definable in $\mathcal{U}_\diamond$.

Since the FO-definable relations in $\mathcal{U}_\diamond$ are exactly the $\diamond$-regular ones, an automatic presentation can be seen as a collection of *automata* describing $\mathcal{A}$ (up to isomorphism).
\[(\mathbb{N}; +) \text{ is in } W\text{-Aut}\]

- Code \(n \in \mathbb{N}\) by its base 2 representation (least significant digit first).
- \(\mathbb{N}\) corresponds to the regular language \(2^*1\).
- The atomic relation \(+\) corresponds to an regular relation \(\text{add}\) over this coding.
- So \((2^*1; \text{add})\) is a word-automatic presentation of \((\mathbb{N}; +)\).
We have seen that the structure with domain $2^*$ and binary relation $x \leq y$ if $[x \sqcap y]_0 \triangleleft_{\text{prefix}} x$ or $[x \sqcap y]_1 \triangleleft_{\text{prefix}} y$ is isomorphic to the rational ordering.
Prefix recognisable graphs are in $W$-Aut

Recall: a graph $(G; E)$ is prefix recognisable if $G \subseteq 2^*$ is regular, and $E$ is a finite union of relations of the form

$$U(V \times W),$$

for $U, V, W \subseteq 2^*$ regular.

**Exercise.** [Blumensath 2002] More generally, every tree interpretable structure is in $W$-Aut.

What about graphs at higher levels of the pushdown hierarchy?
\( \mathbb{R}[0, 1]; + \mod 1 \) is in \( W^\omega - \text{Aut} \)

- Code \( r \in \mathbb{R}[0, 1] \) by its base 2 representation(s) (least significant digit first).
- \( \mathbb{R}[0, 1] \) corresponds to the regular language \( 2^\omega \).
- The atomic relation \( + \mod 1 \) corresponds to a regular relation \( \text{realadd} \) over this coding.
- Equality = corresponds to a regular relation \( \approx \) over this coding; namely \( ((1), (0), (1), (0)) \ast (1) (0) ^\omega \).
- So \( \{0, 1\}^\omega; \text{realadd}, \approx \) is an \( \omega \)-word automatic presentation of \( \mathbb{R}[0, 1]; + \mod 1 \).

Extend this idea to \( \mathbb{R}, + \).
\((\mathbb{N}; \times)\) is in \(T\text{-Aut}\)

A tree \(T_n\) codes the unique factorisation of \(n\) into prime powers
\[ n = p_1^{n_1} \times p_2^{n_2} \times \cdots. \]

- Tree \(T_n\) has domain \(0^*1^*\), and the sequence of labels on the branch \(0^j1^*\) codes \(n_j\) in binary.
- Multiplication then corresponds to addition on the branches.
- This can be done with a tree automaton.

This \(T^\omega\text{-Aut}\) presentation can easily be turned into a \(T\text{-Aut}\) presentation.
**Universal structures**

**Definition.** A structure $\mathcal{U}$ is *universal* for the class $\Diamond-\text{Aut}$, if

- $\mathcal{U}$ is in $\Diamond-\text{Aut}$, and
- every structure in $\Diamond-\text{Aut}$ is FO-interpretable in $\mathcal{U}$.

By definition the structure $\mathcal{U}_\Diamond$ is universal for $\Diamond-\text{Aut}$.

There are other (non-isomorphic) universal structures. Define the numerical predicate $x \mid_2 y$ if $y = xk$ and $x = 2^n$ for some $n, k \in \mathbb{Z}$.

**Theorem.**

- $(\mathbb{N}; +, \mid_2)$ is universal for $\mathbb{W}-\text{Aut}$.
- $(\mathbb{R}; +, \mid_2)$ is universal for $\mathbb{W}^\omega-\text{Aut}$.

Are there natural numerical structures that are universal for $T-\text{Aut}$, $T^\omega-\text{Aut}$?
1-dimensional interpretations suffice

**Proposition.** If $\mathcal{A}$ is FO-interpretable in $\mathcal{U}_\Diamond$, then it is 1-dimensionally FO-interpretable in $\mathcal{U}_\Diamond$.

**Sketch.** Given an $r$-dimensional interpretation, consider a tuple $(a_1, \cdots, a_r)$ from $\mathcal{U}_\Diamond$ representing an element of $\mathcal{A}$. Then $\otimes(\bar{a})$ is a single $\Diamond$ over alphabet $\Gamma = (2 \cup \{\square\})^r$ (the padding symbol $\square$ is required in the cases of finite words and trees).

This is the co-ordinate map of a 1-dimensional FO-interpretation of $\mathcal{A}$ in $\mathcal{U}_\Diamond(\Gamma)$ ($:= \mathcal{U}_\Diamond$ on alphabet $\Gamma$ instead of 2).

Finally, $\mathcal{U}(\Gamma)$ is 1-dimensionally FO-interpretable in $\mathcal{U}(2)$.
EXERCISE. Show that in the word case, the interpretation can be taken 1-dimensional \textit{and injective}. The same holds for the tree case.

PROBLEM. Do injective presentations suffice in the \(\omega\)-word and \(\omega\)-tree cases?
Corollary. A structure is in ♦-Aut if and only if it is interpretable as follows:

\[
\diamond = \begin{cases} 
\text{word} & \text{finite-set interpretable in } (\mathbb{N}, s). \\
\omega\text{-word} & \text{set interpretable in } (\mathbb{N}, s). \\
\text{tree} & \text{finite-set interpretable in } (\{0, 1\}^*, s_0, s_1). \\
\omega\text{-tree} & \text{set interpretable in } (\{0, 1\}^*, s_0, s_1).
\end{cases}
\]