Homework set II

Question 1. Is there a Rényi type theorem for weak-mixing systems? (Recall: Rényi theorem says that $T$ is strong-mixing iff $\mu(T^{-n}A \cap A) \to \mu(A)^2$, $\forall A \in \mathcal{B}$.)

Question 2. Give an alternative proof of the following: if $T$ has no non-constant measurable eigenfunction, then $T$ is weak-mixing.

(Hint: show that if $T$ has no non-constant measurable eigenfunction, then

$$\lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} \left| \langle U_{n}f, f \rangle \right|^{2} \to 0.$$ )

Question 3. Let $R_{\alpha}$ be the circle rotation by an irrational number $\alpha$. (We saw that it is uniquely ergodic.)

1. Show equidistribution property:

$$\lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} \chi_{[a,b]}((R_{\alpha})^{n}(t)) \to (b-a).$$

2. Consider the first digit of $2^{n}$, i.e. 1, 2, 4, 8, 1, 3, 6, ···. What is the density of the first digit being $k$ ($0 \leq k \leq 9$)?

Question 4. Let $X$ be a compact metrizable group.

1. Show that there is a bi-invariant metric on $X$ defining the topology on $X$.

2. Show that $d_{Y}(x) = \min\{d(x, y) : y \in Y\}$ is a non-constant function on $X$ which is constant on each coset of $Y$.

(Reference: Einsiedler-Ward, Chapter 1, 2, 4, 7)