Math 622 take-home final exam due Thursday 11 May at 5:00 pm

Please work independently. Solutions to some of these problems exist in published form, so I ask you to refrain from using the library and other resources, including the Web. You may use the textbook and any result that has been given in class or in the homework. You may also consult with me.

From chapter 6 in the book: 4.6, 5.1, 5.3 (also show that $\sigma_p(A)$ is empty), 5.4.

In the following problems $F$ denotes either of the fields $\mathbb{R}$ or $\mathbb{C}$.

1. Let $H$ be a Hilbert space over $F$ and let $T \in B(H,H)$ be a compact selfadjoint operator. As usual we put $T_\lambda = \lambda I - T$ and $N_\lambda = N_{T_\lambda}$, the $\lambda$-eigenspace of $T$. We also define $N = \text{span}(N_\lambda | \lambda \in \mathbb{R})$, i.e. the subspace of $H$ consisting of finite sums of eigenvectors of $T$. Recall from class that $N$ is a direct sum, $N = \bigoplus_{\lambda \in \mathbb{R}} N_\lambda$, i.e. every $x \in N$ can be written uniquely as a finite sum of eigenvectors, $x = \sum_{k=1}^n x_k$, with $x_k \in N_{\lambda_k}$ and $\lambda_k \neq \lambda_i$ for $k \neq i$.

(a) $\sigma_T(T)$ is nonempty.
(b) $N$ is dense in $H$. (Argue by contradiction, using part (a) and problem 6.5.4.)
(c) $H$ has a Hilbert basis $A = \{ a_i | i \in I \}$ (where $I$ is some index set, possibly infinite) such that $a_i$ is an eigenvector of $T$ for all $i \in I$. (This is the spectral theorem for compact selfadjoint operators. We call $A$ an eigenbasis for $T$.)
(d) Let $\lambda_i$ be the eigenvalue of $a_i$. Then $T x = \sum_{i \in I} \lambda_i (x, a_i) a_i$ for all $x \in H$.
(e) There exists a sequence $(T_n)_{n \geq 1}$ in $B(H,H)$ such that $T_n$ has finite-dimensional range for all $n$ and $\lim_{n \to \infty} T_n = T$ in operator norm.

2. Let $H$ be a Hilbert space over $F$ and let $S$ and $T \in B(H,H)$ be compact selfadjoint operators. Assume that $ST = TS$.

(a) $SN_{T_\lambda} \subseteq N_{T_\lambda}$ for all $\lambda \in \mathbb{R}$.
(b) $H$ has a Hilbert basis $A = \{ a_i | i \in I \}$ such that $a_i$ is an eigenvector of $S$ and of $T$ for all $i \in I$. (We call $A$ a joint eigenbasis for $S$ and $T$. Don’t argue from scratch, but use part (a) and the spectral theorem for a single operator.)

3. Let $H$ be a Hilbert space over $F$ and let $T \in B(H,H)$ be a compact operator.

(a) There exist unique compact selfadjoint operators $T_1$ and $T_2$ such that $T = T_1 + iT_2$.
(b) $T$ is normal (i.e. $TT^* = T^*T$) if and only if $T_1 T_2 = T_2 T_1$.
(c) State and prove a spectral theorem for compact normal operators.

4. Let $H$ be the space of measurable functions $f : \mathbb{R} \to \mathbb{C}$ which are $2\pi$-periodic (i.e. $f(x) = f(x + 2\pi)$ for all $x$) and which satisfy $\int_{-\pi}^\pi |f(x)|^2 \, dx < \infty$, where $dx$ is Lebesgue measure. Then $H$ is a complex Hilbert space with the inner product defined by

$$
(f,g) = \frac{1}{2\pi} \int_{-\pi}^\pi f(x)\overline{g(x)} \, dx
$$

for $f, g \in H$. Let $\hat{\rho} : [-\pi, \pi] \to [0, \infty)$ be a $C^\infty$ function satisfying $\hat{\rho}(x) = 0$ for $|x| \geq 1$, $\int_{-\pi}^\pi \hat{\rho}(x) \, dx = 2\pi$ and $\hat{\rho}(-x) = \hat{\rho}(x)$. (See problem 3.3.1 for an example of such a function.) For $0 < \varepsilon < 1$ put

$\hat{\rho}_\varepsilon(x) = \hat{\rho}(x/\varepsilon)/\varepsilon$ if $|x| \leq \varepsilon$ and $\hat{\rho}_\varepsilon(x) = 0$ if $\varepsilon \leq |x| \leq \pi$. Extend $\hat{\rho}_\varepsilon$ to a $2\pi$-periodic function $\rho_\varepsilon : \mathbb{R} \to [0, \infty)$ and define

$$
K_\varepsilon f(x) = \frac{1}{2\pi} \int_{-\pi}^\pi \rho_\varepsilon(y)f(x-y) \, dy.
$$

(a) Draw a picture of $\rho_\varepsilon$ for a few values of $\varepsilon$. Show $\int_{-\pi}^\pi \rho_\varepsilon(x) \, dx = 2\pi$ for all $0 < \varepsilon < 1$.
(b) $K_\varepsilon f(x) = \frac{1}{2\pi} \int_{-\pi}^\pi \rho_\varepsilon(x-y)f(y) \, dy$ for all $f \in H$. (over)
(c) $K_{\epsilon} : H \to H$ is a compact selfadjoint operator for all $\epsilon > 0$.
(d) $K_{\epsilon} f$ is $C^\infty$ for all $f \in H$.
(e) $\lim_{\epsilon \to 0} \|K_{\epsilon} f - f\| = 0$ in supremum norm if $f \in H$ is continuous.
(f) $\lim_{\epsilon \to 0} \|K_{\epsilon} f - f\| = 0$ in the norm defined by the inner product for all $f \in H$. The space of smooth $2\pi$-periodic functions is dense in $H$.
(g) Is $\lim_{\epsilon \to 0} K_{\epsilon} = I$ in operator norm?