Math 612  homework assignment 7    due 2007-03-29 at 3 pm

1. Show that the Bergman kernel of the open unit disc $D = D(0,1)$ is given by

$$K(z, \zeta) = \frac{1}{\pi(1 - z\zeta)^2}.$$  

2. For any nonempty region $\Omega$ in $\mathbb{C}$ let $HL^2(\Omega) = H(\Omega) \cap L^2(\Omega)$ be the Hilbert space studied in homework 6 and let $K_{\Omega}$ be its Bergman kernel. Let $f: \Omega_1 \rightarrow \Omega_2$ be a biholomorphic map from a region $\Omega_1$ to a region $\Omega_2$. Prove the following assertions.

   (i) The transformation $T: HL^2(\Omega_2) \rightarrow HL^2(\Omega_1)$ defined by

   $$T(\phi)(z) = \phi(f(z))f'(z)$$

   for $\phi \in HL^2(\Omega_2)$ and $z \in \Omega_1$ is a surjective isometry.

   (ii) $K_{\Omega_1}(z, \zeta) = K_{\Omega_2}(f(z), f(\zeta))f'(z)f'(\zeta)$ for all $(z, \zeta) \in \Omega_1 \times \Omega_1$.

   (iii) Suppose $\Omega \neq \mathbb{C}$ is a simply connected region. Let $z_0 \in \Omega$ and define

   $$f(z) = \left( \frac{\pi}{K_{\Omega}(z_0, z_0)} \right)^{\frac{1}{2}} \int_{z_0}^{z} K_{\Omega}(\zeta, z_0) \, d\zeta$$

   for $z \in \Omega$. Then $f(z_0) = 0$, $f'(z_0) > 0$ and $f$ is a biholomorphism from $\Omega$ to the unit disc $D = D(0,1)$. 
