Math 612 homework assignment 2 due 2007-02-08 at 3 pm

You may collaborate on all the problem sets, but you must write up the solutions individually and list your collaborators.

1. Let $\Omega$ be a region in $\mathbb{R}^2$ and $u : \Omega \to \mathbb{R}$ a $C^2$ function (i.e. all the partial derivatives of $u$ of order $\leq 2$ exist and are continuous). The Laplacian of $u$ is

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2},$$

and $u$ is harmonic if $\Delta u = 0$. If $f : \Omega \to \mathbb{C}$ is a $C^2$ function, we write $f = u + iv$ with $u$ and $v$ real and put $\Delta f = \Delta u + i\Delta v$. Show the following facts.

(i) $\Delta f = 4 \frac{\partial}{\partial z} \frac{\partial}{\partial \overline{z}} f$.
(ii) If $f : \Omega \to \mathbb{C}$ is $C^2$ and holomorphic, then it is harmonic. (We shall see later that holomorphic implies $C^2$.)

2. Problems 11.1 and 11.5 from the book.

3. Let $\Omega$ be a region in $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$. Prove that the following two statements are equivalent.

(i) $\text{Ind}(\gamma, 0) = 0$ for every loop $\gamma$ in $\Omega$.
(ii) There is a continuous map $\log : \Omega \to \mathbb{C}$ satisfying $e^{\log z} = z$ for all $z \in \Omega$.

4. Let $\text{GL}(2, \mathbb{C})$ be the set of all complex matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $ad - bc \neq 0$. It is well-known that $\text{GL}(2, \mathbb{C})$ is a group under matrix multiplication (called the general linear group of rank 2). The Möbius transformation associated with $A$ is the map $\phi_A : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ defined by

$$\phi_A(z) = \frac{az + b}{cz + d}.$$ 

Let $G$ denote the collection of all Möbius transformations. Prove the following assertions.

(i) $\phi_A \circ \phi_B = \phi_{AB}$.
(ii) Möbius transformations are continuous and bijective.
(iii) $G$ is a subgroup of the group of all homeomorphisms from $\hat{\mathbb{C}}$ onto itself.
(iv) The map $\phi : \text{GL}(2, \mathbb{C}) \to G$ defined by $\phi(A) = \phi_A$ is a surjective homomorphism. Its kernel is the subgroup consisting of all matrices $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ with $a \neq 0$.
(v) For each triple $\alpha$, $\beta$, $\gamma$ of distinct points in $\mathbb{C}$ there is a unique $g \in G$ such that $g(\alpha) = 0$, $g(\beta) = 1$ and $g(\gamma) = \infty$.

5. A fixed point of a map $f : X \to X$ is an element $x \in X$ satisfying $f(x) = x$. Two elements $g_1$ and $g_2$ of a group $G$ are conjugate if there exists $h \in G$ such that $g_2 = hg_1h^{-1}$. Prove the following assertions.

(i) A Möbius transformation other than the identity has either one or two fixed points.

(over)
(ii) A Möbius transformation has one fixed point if and only if it is conjugate to the translation $z \mapsto z + 1$.

(iii) A Möbius transformation has two fixed points if and only if it is conjugate to a dilation $z \mapsto \lambda z$, where $\lambda \in \mathbb{C} \setminus \{0, 1\}$. 