Math 612  homework assignment 1  due 2007-02-01 at 3 pm

1. Problem 10.6 from the book. The question, In what other discs can this be done? means: state and prove similar results for the disc $D(z_0, r)$ instead of $D(1, 1)$. For what values of $z_0$ and $r$ do your results hold?

2. Show that an open subset $\Omega$ of $\mathbb{R}^n$ is connected if and only if it is path-connected (in the sense that for every $a, b \in \Omega$ there is a piecewise $C^1$ path $\gamma: [0, 1] \to \Omega$ with $\gamma(0) = a$ and $\gamma(1) = b$).

3. Let $\Omega_1$ and $\Omega_2$ be open subsets of $\mathbb{C}$ and let $f: \Omega_1 \to \Omega_2$ and $g: \Omega_2 \to \mathbb{C}$ be real differentiable functions. Let $h = g \circ f$, i.e. $h(z) = g(f(z))$. Derive from the multivariable chain rule that

$$\frac{\partial h}{\partial z} = \frac{\partial g}{\partial w} \frac{\partial f}{\partial z} + \frac{\partial g}{\partial w} \frac{\partial f}{\partial z},$$

$$\frac{\partial h}{\partial \bar{z}} = \frac{\partial g}{\partial w} \frac{\partial f}{\partial \bar{z}} + \frac{\partial g}{\partial w} \frac{\partial f}{\partial \bar{z}}.$$

4. Polar coordinates $(r, \theta)$ on $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ are defined by $x = r \cos \theta$, $y = r \sin \theta$, or equivalently $z = re^{i\theta}$.

   (i) Show that in polar coordinates the Cauchy-Riemann equations take the form

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}, \quad \frac{1}{r} \frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}.$$

   (ii) Let $\Omega_0$ be the region $r > 0$ and $-\pi < \theta < \pi$. Define $\log: \Omega_0 \to \mathbb{C}$ by $\log z = \log r + i\theta$. Show that $\log$ is holomorphic. Find a region $\Omega_1$ such that $\exp: \Omega_1 \to \Omega_0$ is bijective and $\log$ is its inverse.

5. Define $f: \mathbb{C} \to \mathbb{C}$ by $f(x + iy) = \sqrt{|xy|}$.

   (i) Show that $f$ satisfies the Cauchy-Riemann equations at the origin, but is not holomorphic at the origin.

   (ii) Explain why this does not contradict the theorem that $f$ is holomorphic at $a$ if and only if it is real differentiable at $a$ and $\frac{\partial f}{\partial \bar{z}}(a) = 0$.

6. Let $f$ be holomorphic in a region $\Omega$. Assume that one of the following assumptions holds,

   (i) $\text{Re } f$ is constant, or
   (ii) $\text{Im } f$ is constant, or
   (iii) $|f|$ is constant,

and conclude that $f$ is constant.