1 (15 points). Let $f_n$ be the integrand. Note that $\lim_{n \to \infty} f_n(x) = e^{-3x}$ for all $x$ and that $|f_n(x)| \leq e^{-2x}$ for all $x$. Since $e^{-2x}$ is in $L^1([0, \infty])$, the DCT applies and we get

$$\lim_{n \to \infty} \int_0^\infty \frac{e^{-(3+n^2) \sin x}}{1+n^{-2}x^2} \, dx = \int_0^\infty e^{-3x} \, dx = \frac{1}{3}.$$ 

2 (35 points). (a) $f$ is continuous and has compact support, so is uniformly continuous. Let $\varepsilon > 0$ and choose $\delta > 0$ such that $|f(z) - f(-y)| \leq \varepsilon$ when $|z + y| \leq \delta$. Letting $x = y + z$ we get $|f(x) - f(-y)| \leq \varepsilon$ when $|x| \leq \delta$. If $|x| \leq 1$, then $f(x) - f(-y) = 0$ for $-2 \leq y \leq 2$, because $f$ is supported on $[-1,1]$. Hence, for $|x| \leq \min(1,\delta)$,

$$|h(x) - h(0)| = \left| \int_{-2}^2 (f(x-y) - f(-y))g(y) \, dy \right| \leq \varepsilon \int_{-2}^2 |g(y)| \, dy,$$

which proves $\lim_{x \to 0} h(x) = h(0)$. An alternative proof uses the DCT.

(b) $h$ is still continuous at 0. One possible proof is to note that

$$h(x) = \int_{-\infty}^\infty f(y)g(x-y) \, dy$$

by translation-invariance of Lebesgue measure, and to apply the argument of part (a) with $f$ and $g$ reversed. Another proof is to define the Banach space

$$E = \{ f \in L^1 \mid \text{ess} \sup f \subseteq [-1,1] \}$$

(with $L^1$ norm) and to observe that the assignment $f \mapsto h$ is a bounded linear map $T_f : E \to L^\infty$. Now choose a sequence of continuous functions $(f_n)$ in $E$ converging to $f$. Then $Tf_n \to Tf$ in $L^\infty$ norm. By part (a) each $Tf_n$ is continuous, so $Tf$ is continuous.

3 (15 points). The hypothesis says that $\lim_{n \to \infty} f_n = 0$ in $L^3([0,1])$. Let $p = 3$ and $q = 3/2$. Then $p$ and $q$ are complementary and $g(x) = 1/\sqrt{x}$ is in $L^q$, because $g(x)^q = x^{-3/4}$ and $-3/4 > -1$. Hence by Hölder

$$0 \leq \lim_{n \to \infty} \int_0^1 f_n(x)g(x) \, dx \leq \lim_{n \to \infty} \int_0^1 |f_n(x)g(x)| \, dx \leq \lim_{n \to \infty} \|f_n\|_p \|g\|_q = 0.$$ 

4 (35 points). Let $H$ be the real Hilbert space $L^2([-1,1], \mathbb{R})$ and $M$ the subspace spanned by the functions $g_1(x) = 1$ and $g_2(x) = x$. The hypotheses say that $||f||^2 = 21$ and $(f,g_1) = 6$. Let $g = \text{P}_Mf$; then $g$ can be any element of $M$ satisfying $||g||^2 \leq 21$. (Reason: let $g \in M$ satisfy $||g||^2 \leq 21$. Let $f = g + g_3$, where $g_3$ is any element of $M^\perp$ satisfying $||g_3||^2 = 21 - ||g||^2$; then $||f||^2 = 21$ and $(f,g_1) = (g,g_1) + (g_3,g_1) = 6$.) Since $g_1 \perp g_2$, we have $g = a_1g_1 + a_2g_2$ with $a_1 = (g,g_1)/(g_1,g_1)$ and $a_2 = (g,g_2)/(g_2,g_2)$. Hence, by Pythagoras,

$$||g||^2 = \frac{||g_1||^2}{||g_1||^2} + \frac{||g_2||^2}{||g_2||^2}.$$ 

Also $(g,g_1) = (f,g_1) = 6$, $(g_1,g_1) = 2$, $(g_2,g_2) = 2$, so

$$\int_{-1}^1 xf(x) \, dx = ||f||^2 = ||g||^2 \leq \frac{2}{3}(21 - 18) = 2.$$ 

Since $||g||^2$ can attain any value between 0 and 21, this is the only constraint on the integral $\int_{-1}^1 xf(x) \, dx$. 

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Summary solutions