Math 433  practice problems  6 December 2005

Problems 13.8 and 13.9 from the book.

1. Let $F$ be a field and $A \in M_n(F)$. The trace of $A$ is defined by $\text{tr} A = \sum_{i=1}^n a_{ii}$. Let $V$ be an $n$-dimensional vector space over $F$ and let $f : V \to V$ be linear. The trace of $f$ is defined by $\text{tr} f = \sum_{i=1}^n a_{ii}$, where $A = (a_{ij})$ is the matrix of $f$ with respect to an ordered basis of $V$.

(i) $\text{tr}(AB) = \text{tr}(BA)$ for $A, B \in M_n(F)$.
(ii) The trace of $f$ is well-defined, i.e. independent of the choice of basis.
(iii) $\text{tr}(f \circ g) = \text{tr}(g \circ f)$ for $f, g \in \mathcal{L}(V, V)$.
(iv) Define $c_i \in F$ by $\det(x \text{id}_V - f) = \sum_{i=0}^n c_i x^i$, i.e. $c_i$ is the $i$-th coefficient of the characteristic polynomial $\chi_f(x) = \det(x \text{id}_V - f)$. Then $c_n = 1$, $c_{n-1} = -\text{tr} f$, $c_0 = (-1)^n \det f$.
(v) Suppose $F = \mathbb{R}$ or $\mathbb{C}$. Then $e^{\text{tr} A} = \det \exp A$.

2. Suppose $V$ is a real vector space of odd dimension $n$. Let $f, g : V \to V$ be linear maps satisfying $f \circ g = g \circ f$. Then $f$ and $g$ have a common eigenvector.

3. Let $F$ be any field and let $A \in M_n(F)$ and $B \in M_m(F)$. If $A$ and $B$ are diagonalizable, then so is $A \otimes B$.

4. Let $V$ be a vector space over a field $F$ and let $f$ be an endomorphism of $V$ satisfying $f^2 = f$.

(i) The only possible eigenvalues of $f$ are 0 and 1.
(ii) $f$ is diagonalizable (whether or not $V$ is finite-dimensional).

5. Find the solution of the system

$x' = Ax$

satisfying the initial condition $x(0) = (1, 0, 1)^T$, where $A$ is the real matrix

$$A = \begin{pmatrix} -14 & 6 & 15 \\ -4 & 0 & 5 \\ -8 & 4 & 8 \end{pmatrix}.$$