1 (20 points).  
(a) \(d(x, b) - d(x, a) \leq d(a, b)\) and \(d(x, a) - d(x, b) \leq d(a, b)\) by triangle inequality, so \(|f_b(x)| = |d(x, b) - d(x, a)| \leq d(a, b)\).

(b) \(|f_b(y) - f_b(x)| = |d(y, b) - d(y, a) - d(x, b) + d(x, a)| \leq |d(y, b) - d(x, b)| + |d(y, a) - d(x, a)| \leq 2d(y, x)\) by using triangle inequality twice. So let \(\epsilon > 0\) and \(\delta = \epsilon/2\). Then for all \(x, y \in X\) such that \(d(x, y) \leq \delta\) we have \(|f_b(y) - f_b(x)| < \epsilon\).

(c) \(|f_b(x) - f_c(x)| = |d(x, b) - d(x, a) - d(x, c) + d(x, a)| \leq |d(x, b) - d(x, c)| \leq d(b, c)\) by triangle inequality, so \(\|f_b - f_c\|_{\text{sup}} \leq d(b, c)\). On the other hand, if \(x = b\) or \(x = c\), then \(|f_b(x) - f_c(x)| = d(b, c)\), so \(\|f_b - f_c\|_{\text{sup}} = d(b, c)\).

2 (15 points).  
(a) \(O_n\) are open because \(f, f_n\) continuous. \(O_n \subseteq O_{n+1}\) because \(f_n \leq f_{n+1}\). Let \(x \in X\). Choose \(n\) such that \(f(x) - f_n(x) < \epsilon\) (possible because \(f(x) = \lim_{n \to \infty} f_n(x)\) and \(f_n(x) \leq f(x)\)). Then \(x \in O_n\), so \(O_n\) covers \(X\).

(b) Let \(\epsilon > 0\). Choose \(N\) such that \(X = O_N\), where \(O_N\) is as in previous part (possible because \(X\) compact and the \(O_n\)s are nested). Then \(f(x) - f_n(x) \leq f(x) - f_N(x) < \epsilon\) for all \(x\) and all \(n \geq N\), so \(\|f - f_n\|_{\text{sup}} \leq \epsilon\) for all \(n \geq N\), i.e. \(f_n \to f\) uniformly.

3 (30 points).  
(a) By mean value theorem \(\phi_n(x, y) = \partial f(x, y)/\partial y\) for certain \(\eta\) between \(y\) and \(y + h_n\). Hence \(|\phi_n(x, y)| \leq |g(x)|\).

(b) \(f(x, y)\) is integrable, hence \(g(x)\) is integrable, for each \(y\), hence so is \(\phi_n\). By definition \(\partial f(x, y)/\partial y = \lim_{n \to \infty} \phi_n(x, y)\) (pointwise limit), so \(\partial f(x, y)/\partial y\) also measurable.

(c) \[\frac{F(y + h_n) - F(y)}{h_n} = \int_R \phi_n(x, y) \, dx\]

by linearity. We know \(|\phi_n(x, y)| \leq g(x)|\), where \(g\) is Lebesgue integrable, and \(\partial f(x, y)/\partial y = \lim_{n \to \infty} \phi_n(x, y)\) pointwise, so by dominated convergence theorem \(\partial f(x, y)/\partial y\) is Lebesgue integrable and

\[\int_R \frac{\partial f}{\partial y}(x, y) \, dx = \lim_{n \to \infty} \int_R \phi_n(x, y) \, dx = \lim_{n \to \infty} \frac{F(y + h_n) - F(y)}{h_n}\]

Since this holds for all sequences \(h_n \to 0\), we conclude that \(F\) is differentiable and that \(F'(y) = \int_R \partial f(x, y)/\partial y \, dx\).

4 (35 points).  
(a) \(f\) is measurable because it’s the characteristic function of a measurable set, and \(|f|^2 = f\) is integrable because \([-\delta, \delta]\) has finite measure, so \(f \in L^2\).

(b) \(c_0(f) = \delta/\pi\) and for \(n \neq 0\),

\[c_n(f) = (f, e_n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx = \frac{1}{2\pi} \int_{-\delta}^{\delta} f(x) e^{-inx} \, dx\]

\[= \frac{1}{2\pi} \left[ -\frac{1}{in} e^{-inx} \right]_{x=-\delta}^{x=\delta} = \frac{\sin(n\delta)}{n\pi}.\]

(c) \(|f|^2 = 2\delta/2\pi = \delta/\pi\) and \(\sum_{n=-\infty}^{\infty} |c_n|^2 = \pi^{-2}\delta^2 + \pi^{-2} \sum_{n \neq 0} n^{-2} \sin^2(n\delta)\), so by Parseval

\[\frac{\delta}{\pi} = \frac{\delta^2}{\pi^2} + \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin^2(n\delta)}{n^2} \Rightarrow \sum_{n=1}^{\infty} \frac{\sin^2(n\delta)}{n^2} = \frac{\pi - \delta}{2}.\]

(d) \[\frac{\pi^2}{\delta} = \sum_{n=1}^{\infty} \frac{\sin^2(n\pi/2)}{n^2} = \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2}.\]
(e) Each \( f_m \) is a monotone limit \( \lim_{N \to \infty} f_{mN} \), where \( f_{mN} \) is the simple function

\[
\sum_{n=1}^{N} f\left(\frac{n}{m}\right) \chi_{\left((n-1)/m, n/m\right)},
\]

so \( f_m \) is measurable. Hence by monotone convergence theorem

\[
\int_{0}^{\infty} f_m(x) \, dx = \sum_{n=1}^{\infty} \frac{1}{m} f\left(\frac{n}{m}\right).
\]

Now let \( x \geq 0 \). Let \( n \) be the unique positive integer such that \( (n-1)/m \leq x < n/m \), i.e. \( n-1 = \lfloor mx \rfloor \) (integral part). Then \( f_m(x) = f(\lfloor mx \rfloor/m + 1/m) \). Observe \( \lim_{m \to \infty} \lfloor mx \rfloor/m = x \) (because \( |\lfloor mx \rfloor/m - x| = |(\lfloor mx \rfloor - mx)/m| \leq 1/m \)), so

\[
\lim_{m \to \infty} f_m(x) = f(x)
\]

because \( f \) continuous.

Note also \( |f_m| \leq g \) where \( g(x) = x^2 \) for \( x > 1 \) and \( g(x) = 1 \) for \( x \leq 1 \). This \( g \) is integrable, so by dominated convergence theorem

\[
\int_{0}^{\infty} f(x) \, dx = \lim_{m \to \infty} \int_{0}^{\infty} f_m(x) \, dx = \lim_{m \to \infty} \sum_{n=1}^{\infty} \frac{1}{m} f\left(\frac{n}{m}\right) = \frac{\pi}{2},
\]

where the last equality follows from (c).