Math 414 Spring 2005

Comment on double sequences

Let $x_{mn} = m/(m + n)$. Then

$$
\lim_{n \to \infty} \lim_{m \to \infty} x_{mn} = 1,
\lim_{m \to \infty} \lim_{n \to \infty} x_{mn} = 0,
$$

so one has to be careful when dealing with repeated limits. When is interchanging the order allowed? We will leave the following result as an exercise.

1. **Theorem.** Suppose $x_{mn}$ is of the form $f_n(x_m)$, where $\{f_n\}$ is a sequence of continuous functions on $\mathbb{R}$ converging uniformly to a function $f$ and $\{x_m\}$ is a sequence of reals converging to $x$. Then

$$
\lim_{n \to \infty} \lim_{m \to \infty} x_{mn} = \lim_{m \to \infty} \lim_{n \to \infty} x_{mn} = f(x).
$$

The next result is more useful in Lebesgue’s theory of integration.

2. **Theorem.** Suppose $(x_{mn})_{m,n \geq 0}$ is monotone nondecreasing in each variable separately, i.e. $m_1 \leq m_2$ implies $x_{m_1,n} \leq x_{m_2,n}$ for all $n$ and $n_1 \leq n_2$ implies $x_{m,n_1} \leq x_{m,n_2}$ for all $m$. Let $L = \sup\{x_{mn} \mid m, n \geq 0\} \in \mathbb{R} \cup \{\infty\}$. Then

$$
\lim_{n \to \infty} \lim_{m \to \infty} x_{mn} = \lim_{(m,n) \to (\infty,\infty)} x_{mn} = \lim_{m \to \infty} \lim_{n \to \infty} x_{mn} = L.
$$

**Proof.** For each $n$ let $L_n = \sup\{x_{mn} \mid m \geq 0\} = \lim_{m \to \infty} x_{mn}$. If $n_1 \geq n_2$ then $x_{m,n_1} \leq x_{m,n_2}$ and thus

$$
L_{n_1} = \lim_{m \to \infty} x_{mn_1} \leq \lim_{m \to \infty} x_{mn_2} = L_{n_2},
$$

so $(L_n)$ is monotone nondecreasing. Let $M = \sup\{L_n \mid n \geq 0\} = \lim_{n \to \infty} L_n$. Then for all $m$ and $n$, $x_{mn} \leq L_n \leq M$, so $M$ is an upper bound for $x_{mn}$, so $M \geq L$. Let $\epsilon > 0$. Choose $n$ such that $M - L_n < \epsilon/2$ and choose $m$ such that $L_n - x_{mn} < \epsilon/2$; then $M - x_{mn} < \epsilon$, so $M \leq L$. This proves

$$
\lim_{n \to \infty} \lim_{m \to \infty} x_{mn} = \lim_{n \to \infty} L_n = M = L.
$$

By interchanging the roles of $m$ and $n$ we see that $\lim_{m \to \infty} \lim_{n \to \infty} x_{mn}$ is also equal to $L$. To finish the proof it is enough to show that $\lim_{(m,n) \to (\infty,\infty)} x_{mn} = L$. Let $\epsilon > 0$. Choose $N_1$ such that $L - L_n < \epsilon/2$ for $n \geq N_1$. Now choose $N_2$ such that $L_{N_1} - x_{mN_1} < \epsilon/2$ for $m \geq N_2$. Put $N = \max\{N_1, N_2\}$ and let $m, n \geq N$. Then $x_{mn} \geq x_{N_2N_1}$, so

$$
L - x_{mn} \leq L - x_{N_2N_1} = L - L_{N_1} + L_{N_1} - x_{N_2N_1} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,
$$

so $\lim_{(m,n) \to (\infty,\infty)} x_{mn} = L$. □