Math 414 Spring 2005
Homework Assignment no. 9
Due Thursday 5 May

14.3.5: 8; 14.4.4: 1, 4; notes on Lebesgue integration: 7, 10, 11, 12.

1. Let $X$ be a set and $f : X \to [0, \infty)$ a function. We say that $f$ is summable if

$$s = \sup \left\{ \sum_{x \in X'} f(x) \bigg| X' \text{ is a finite subset of } X \right\}$$

is finite. If this is the case, then $s$ is the sum of $f$. Prove that if $f$ is summable, then $f(x) = 0$ for all except at most a countable number of $x \in X$. (In other words, uncountable “series” of positive terms always diverge.)

2. For $n \geq 1$ define $f_n \in C([0, 1], \mathbb{R})$ by $f_n(x) = 1/\sqrt{x}$ if $1/n \leq x \leq 1$ and $f_n(x) = \sqrt{n}$ if $0 \leq x \leq 1/n$. Prove that $\{f_n\}$ is Cauchy in the $L^1$-norm, but not in the sup-norm. Conclude that $C([0, 1], \mathbb{R})$ is not complete in the $L^1$-norm. Show in a similar way that $C([0, 1], \mathbb{R})$ is not complete in the $L^p$-norm for any $p \geq 1$. 