There are more problems here than many of you can do in fifty minutes, but they are representative of the type of problems you might expect on the exam. On the exam you may use a one-sided letter-size crib sheet, but no books, notes, or calculators.

From notes: 2.7, 2.13, 2.14. (These problems will also be on the next homework assignment.)

1. Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a smooth function and define \( g(x) = \sin(f(x)^2) \). Find the 1-form \( dg \).

2. Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be a smooth function and assume \( f(x,y) = -f(y,x) \). Show that
\[
\frac{\partial f}{\partial x}(a,b) = -\frac{\partial f}{\partial y}(b,a).
\]

3. Thermodynamicists like to use rules such as
\[
\frac{\partial y}{\partial x} \frac{\partial x}{\partial y} = 1.
\]
Explain the rule and show that it is correct. (Assume that the variables are subject to a relation \( F(x,y) = 0 \) defining functions \( x = f(y), \ y = g(x) \), and apply the multivariable chain rule.) Similarly, explain why
\[
\frac{\partial y}{\partial z} \frac{\partial z}{\partial x} \frac{\partial x}{\partial y} = -1.
\]
Naively cancelling numerators against denominators gives the wrong answer!

4. Let \( \alpha = x_1 \, dx_2 + x_3 \, dx_4, \beta = x_1 x_2 \, dx_3 \, dx_4 + x_3 x_4 \, dx_1 \, dx_2 \) and \( \gamma = x_2 \, dx_1 \, dx_3 \, dx_4 \) be forms on \( \mathbb{R}^4 \). Calculate
   
   (a) \( \alpha \beta, \alpha \gamma \);
   
   (b) \( d\beta, d\gamma \);
   
   (c) \( \star \alpha, \star \gamma \).

5. Suppose that \( f : \mathbb{R}^n \to \mathbb{R} \) is a smooth function satisfying
\[
f(t^{a_1} x_1, t^{a_2} x_2, \ldots, t^{a_n} x_n) = t^k f(x)
\]
for all \( x \). Here \( k \) is a positive constant and \( a_1, a_2, \ldots, a_n \) are arbitrary real constants. Deduce from the chain rule that
\[
\sum_{i=1}^n a_i x_i \frac{\partial f}{\partial x_i}(x) = kf(x).
\]