Toric arrangements that come from graphs
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Toric arrangements

- Layman’s terms: lines on a donut.
- Studied in connection to Kostant partition functions (De Concini-Procesi ‘05), arithmetic matroids (Moci ‘12), arithmetic Tutte polynomial (D’Adderio-Moci ‘13), etc.
- This talk is about toric arrangements that are built from graphs.
Motivation

- The current study of toric graphic arrangements is mainly focused on the case of the standard torus.
- We study graphic arrangements on two other types of tori, the coweight torus and the coroot torus.
- We will see that these two arrangements tell us new things about the acyclic orientations of the input graph.
Tori

Let $V$ be a real vector space.

A lattice $L$ is the integer-span of a basis of $V$. The associated torus is the quotient $T := V/L$. 
Graphic arrangements

Let $G$ be a simple connected graph. 
$\mathcal{A}(G)$ is called the linear graphic arrangement.
$\widetilde{\mathcal{A}}(G)$ is called the affine graphic arrangement.
Toric graphic arrangements, example 1

$L_1$

$\tilde{A}(K_3)$

$\overline{A}(K_3, L_1)$
Toric graphic arrangements, example 2

$L_1$
Toric graphic arrangements, example 2
Toric graphic arrangements, example 2

$\tilde{A}(K_3)$

$L_2$

$\overline{A}(K_3, L_2)$
Root system of type A

\[ V_n := \{ x \in \mathbb{R}^n \mid x_1 + \cdots + x_n = 0 \} \text{ (ambient space)}; \]
\[ A_{n-1} := \{ e_i - e_j \mid 1 \leq i < j \leq n \} \text{ (root system of type A)}; \]
\[ \widehat{\mathbb{Z}}A_{n-1} := \mathbb{Z}\{1/n(e_1 + \cdots + e_n) - e_i \mid 1 \leq i \leq n\} \text{ ((co)weight lattice)}; \]
\[ \mathbb{Z}A_{n-1} := \mathbb{Z}\{e_i - e_j \mid 1 \leq i < j \leq n\} \text{ ((co)root lattice).} \]
One graph, three toric arrangements

- The **standard arrangement**: \( V = \mathbb{R}^n, L = \mathbb{Z}^n \).
- The **coweight arrangement**: \( V = V_n, L = \hat{\mathbb{Z}A_{n-1}} \).
- The **coroot arrangement**: \( V = V_n, L = \mathbb{Z}A_{n-1} \).
Toric chambers

\[ \overline{\mathcal{A}}(K_3, \mathbb{Z}A_2) \]

\[ \overline{\mathcal{A}}(K_3, \mathbb{Z}A_2) \]
Acyclic orientations
Recall the bijection of Greene and Zaslavsky ('83):
Chambers of $\mathcal{A}(G)$ ↔ Acyclic orientations of $G$
Coweight Voronoi cells

$\mathbb{Z}A_2$

$A(K_3)$
Two orientations that are projected to the same toric chamber are Voronoi equivalent.
Two orientations that are projected to the same toric chamber are \textit{Voronoi equivalent}. 
Coroot Voronoi cells

$\mathbb{Z}A_2$

$\mathcal{A}(K_3)$
Coroot Voronoi relation

No two distinct acyclic orientations are Voronoi equivalent.
Coroot Voronoi relation

No two distinct acyclic orientations are Voronoi equivalent.
Combinatorial description for coweight Voronoi equivalence

The relation is **source-to-sink** flip.

![Diagram showing the source-to-sink flip]

- Studied by
  - Mosesjan (‘72) and Pretzel (‘86) in combinatorics;
  - Eriksson and Eriksson (‘09), and Speyer (‘09) in connection to conjugacy of Coxeter elements;
  - Develin, Macauley and Reiner (‘16) in the context of toric arrangements.

- It also arises in connection to sandpile groups and chip-firing.
Combinatorial description for coroot Voronoi equivalence

The relation has several equivalent descriptions.

One is source-sink exchange.

Another one is \( n \)-step source-to-sink flip.
THANK YOU!