413: PROBLEM SET 5. DUE THURSDAY 6 MARCH

(1) Let \( X \subset \mathbb{R} \). A point \( x \in X \) is called an isolated point of \( X \) if there exists an open set \( U \) such that \( U \cap X = \{x\} \). Can an open set have an isolated point? Can a closed set have one?

(2) Let \( X \subset \mathbb{R} \) and \( x \in X \) be an isolated point. Let \( f : X \to \mathbb{R} \) be any function. Show that \( f \) is continuous at \( x \).

(3) Consider the following collection of open intervals in \( \mathbb{R} \):
\[
\mathcal{U} = \{(q - \frac{1}{n}, q + \frac{1}{n}) : q \in \mathbb{Q}, n \in \mathbb{N}\}.
\]
(a) Prove that \( \mathcal{U} \) is countable.
(b) Prove that every open subset of \( \mathbb{R} \) may be expressed as a union of intervals from \( \mathcal{U} \).
(c) Let \( \mathcal{V} \) be the collection of all open subsets of \( \mathbb{R} \). Prove that \( |\mathcal{V}| = |\mathbb{R}| \). (Hint: you may use the fact, proved in lectures, that \( |\mathbb{R}| = |\mathcal{P}(\mathbb{N})| \).)

(4) If \( X \subset \mathbb{R} \), prove that the closure of \( X \) is equal to the intersection of all the closed sets containing \( X \).

(5) Recall that an interval in \( \mathbb{R} \) is a subset \( I \) of \( \mathbb{R} \) such that for all \( x, y \in I \) and all \( z \in \mathbb{R} \) with \( x < z < y \), we have \( z \in I \). Prove that if \( I \) is a bounded interval then \((x, y) \subset I\) where \( x = \inf I \) and \( y = \sup I \). Deduce that \( I \) must be one of the following four intervals: \((x, y), (x, y], [x, y), [x, y]\).

(6) Section 3.2.3 #6.

(7) Section 3.3.1 #6.

(8) Section 3.3.1 #8.

(9) Let \( D \subset \mathbb{R} \) and \( f : D \to \mathbb{R} \) be a function. Give a precise definition of the statement
\[
\lim_{x \to \infty} f(x) = \infty.
\]
Show using your definition that \( \lim_{x \to \infty} f(x) = \infty \) where \( f : \mathbb{R} \to \mathbb{R} \) is the function \( f(x) = x^2 \).