413: PROBLEM SET 4. DUE THURSDAY 28 FEBRUARY

(1) Let \( \{x_n\} \) be a Cauchy sequence of rational numbers. Since \( \mathbb{Q} \subset \mathbb{R} \), we may regard \( \{x_n\} \) as a Cauchy sequence of real numbers. Since every Cauchy sequence of real numbers converges, this sequence has a limit. Prove that this limit is the real number defined as the equivalence class of the original sequence \( \{x_n\} \).

(2) Since \( \mathbb{Q} \) is a countable set, we may write down its elements as a sequence \( q_1, q_2, q_3, \ldots \) (this is called an enumeration of \( \mathbb{Q} \)). What are the limit points of such a sequence? Does the set of limit points depend on the choice of enumeration?

(3) Let \( \{x_n\} \) be any sequence of real numbers and let \( \{a_n\} \) be a sequence which converges to 0. Prove that the set of limit points of the sequence \( \{x_n + a_n\} \) is the same as the set of limit points of \( \{x_n\} \).

(4) Find all the limit points in \( \mathbb{R} \cup \{\pm\infty\} \) of the following sequences. Identify the limsup and liminf.
   (a) \( \{(-1)^n + \frac{1}{n} + 2^{-n}\} \).
   (b) \( \{(-2)^n\} \).

(5) Section 3.1.3, #5.

(6) Section 3.1.3, #9.

(7) A subset \( S \subset \mathbb{R} \) is an interval if for all \( x, y \in S \) and for all \( z \in \mathbb{R} \), if \( x < z < y \) then \( z \in S \). Prove that the intersection of an arbitrary collection of intervals is always an interval.

(8) Section 3.2.3 #1.

(9) Section 3.2.3 #7. (Note: limit-point is here a synonym for cluster point; see p. 92 of the textbook.)