(1) Let $A \subset \mathbb{R}$. A point $x \in A$ is called isolated if it is not a cluster point of $A$.

(a) Can an open set have an isolated point? Can a closed set have one?

(b) Give an example of a countable set with no isolated points.

(2) Section 3.3.1 # 8.

(3) Section 4.2.4 # 3. (Recall that an interval is, by definition, a subset $I$ of $\mathbb{R}$ such that for all $x, y \in I$ and all $z \in \mathbb{R}$ with $x < z < y$, we have $z \in I$.)

(4) In this question, we will show that every positive real number has an $n^{th}$ root.

(a) Let $x \in (0, \infty)$ and $n \in \mathbb{N}$. Show that there exist $\alpha, \beta \in \mathbb{R}$ with $\alpha^n < x < \beta^n$.

(b) Show that there exists $y \in \mathbb{R}$ with $x = y^n$.

(c) For $x \in [0, \infty)$, show that there exists a unique $y \in [0, \infty)$ with $x = y^2$. We denote this $y$ by $\sqrt{x}$.

(d) Define $f : [0, \infty) \to \mathbb{R}$ by $f(x) = \sqrt{x}$. Show that $f$ is a continuous function.

(5) Two monasteries, $A$ and $B$, are joined by exactly one path $AB$ which is 20 miles long. One morning, Brother Albert (a monk) sets out from monastery $A$ at 9 am, arriving at monastery $B$ at 9 pm. The next day, he sets out from monastery $B$ at 9 am, arriving at monastery $A$ at 9 pm. On both journeys, he may have stopped to rest, or even walked backwards for some of the time.

(a) Prove that there is a point $x$ on the path $AB$ such that Brother Albert was at $x$ at exactly the same time on both days. (Hint: let $f_i(t)$ be the distance of Brother Albert from $A$ at time $t$ on day $i$, $i = 1, 2$. Apply the Intermediate Value Theorem to a suitable combination of $f_1$ and $f_2$.)

(b) Another monk, Brother Gilbert, has been dabbling in forbidden knowledge. Once per day, by snapping his fingers, he can instantaneously teleport himself to any point within a 3 ft. radius of his current location. Suppose Brother Gilbert makes the same journey as Brother Albert. Does the conclusion from part (a) still hold?
(6) Let \( f : \mathbb{R} \to \mathbb{R} \) be a function. Let \( a > 0 \). We say that \( f \) is \textit{periodic with period} \( a \) if
\[
f(x + a) = f(x)
\]
for all \( x \in \mathbb{R} \).

Suppose \( f : \mathbb{R} \to \mathbb{R} \) is periodic with period \( a \) and define
\[
g(x) = f(1/x)
\]
for \( x > 0 \).

(a) Show that for all \( x > 0 \), we have
\[
f([x, x + a]) = g\left(\left[\frac{1}{x + a}, \frac{1}{x}\right]\right).
\]

(b) Suppose \( f \) is not constant. Show that \( g \) is not uniformly continuous on \((0, \infty)\).

\[\textbf{Remark:} \quad \text{In particular, taking } f(x) = \sin(x) \text{ and } a = 2\pi, \text{ we see that } \sin(1/x) \text{ is not uniformly continuous.}\]