(1) **(20 marks.)** Let \( \{x_n\} \) and \( \{y_n\} \) be sequences of real numbers. Let \( L \in \mathbb{R} \).

(a) Explain what it means to say \( \lim_{n \to \infty} x_n = L \).

It means that for all \( \varepsilon > 0 \) there exists \( N \in \mathbb{N} \) such that if \( n > N \) then \( |x_n - L| < \varepsilon \).

(b) Explain what is meant by \( \limsup_{n} y_n \).

One definition: \( \limsup_{n} y_n \) is the supremum of the set of limit-points (limits of subsequences) of the sequence \( \{y_n\} \). Another definition: \( \limsup_{n} y_n \) is the limit of the sequence \( \{\sup_{k \geq n} y_k\} \) as \( n \to \infty \).

(c) Show that if \( \lim_{n \to \infty} x_n = L \) then \( \lim_{n \to \infty} |x_n| = |L| \).

Suppose \( \lim_{n \to \infty} x_n = L \). Let \( \varepsilon > 0 \). Then there exists \( N \in \mathbb{N} \) such that if \( n > N \) then \( |x_n - L| < \varepsilon \). If \( n > N \) then \( ||x_n| - |L|| \leq |x_n - L| < \varepsilon \).

(d) Suppose \( \limsup_{n} y_n = L \). Is it necessarily true that \( \limsup_{n} |y_n| = |L| \)? Explain your answer.

No. For example, take the sequence \( 0, -1, 0, -1, \ldots \) for \( \{y_n\} \). Then \( \limsup_{n} y_n = 0 \) but \( \limsup_{n} |y_n| = 1 \).

(2) **(20 marks)** A real number \( \alpha \) is said to be **algebraic** if for some \( n \in \mathbb{N} \) there is a polynomial \( f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \) of degree \( n \) with \( a_i \in \mathbb{Q} \) for all \( i \), and with \( f(\alpha) = 0 \). (In this case, we say that \( \alpha \) is a root of \( f \).) If \( \alpha \) is not algebraic, it is said to be **transcendental**.

(a) Show that the set of all algebraic numbers is countable. (You may use without proof the fact that a polynomial of degree \( n \) has at most \( n \) roots.)
The set of all algebraic numbers is the union

\[ \bigcup_{n \geq 1} \bigcup_{p \in P_n} (\text{roots of } p). \]

Where \( P_n \) denotes the set of all polynomials with rational coefficients of degree \( n \). The set \( P_n \) is in bijection with \( \mathbb{Q}^n \), a countable set. We see that the set of algebraic numbers is a countable union of countable sets, so it is countable.

(b) Show that there exists a transcendental number.

Since \( \mathbb{R} \) is uncountable, it cannot be equal to the set of algebraic numbers. So there must be a real number which is not algebraic.

(c) Now consider the expression \( g(x) = \sum_{n=1}^{\infty} x^n! \). Show that the series defines a \( C^\infty \) function \( g : (-1, 1) \to \mathbb{R} \). [Remark: the number \( g(1/10) \) is known to be transcendental. Do not prove this!]

This is a power series \( g(x) = \sum a_n x^n \) with coefficients

\[ a_n = \begin{cases} 
1 & n = k! \\
0 & \text{otherwise.}
\end{cases} \]

We see that \( \limsup_n |a_n|^{1/n} = 1 \) and so the power series has radius of convergence 1. Therefore, by theorems on power series, it defines a \( C^\infty \) function on the interval \( (-1, 1) \).

(3) (20 marks) Let \( f : \mathbb{R} \to \mathbb{R} \) be a function.

(a) State what it means for \( f \) to be uniformly continuous on \( \mathbb{R} \).

It means that for all \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that if \( x, y \in \mathbb{R} \) and \( |x - y| < \delta \) then \( |f(x) - f(y)| < \varepsilon \).

(b) State the Mean Value Theorem.

Suppose \( f \) is continuous on \( [a, b] \) and differentiable on \( (a, b) \). Then there exists \( x \in (a, b) \) with

\[ f'(x) = \frac{f(b) - f(a)}{b - a}. \]

(c) Suppose that \( f : \mathbb{R} \to \mathbb{R} \) is a differentiable function and that the derivative \( f' \) is bounded. Show that \( f \) is uniformly continuous on \( \mathbb{R} \).
Let \( f: \mathbb{R} \to \mathbb{R} \) be differentiable and suppose there exists \( M > 0 \) with \( |f'(x)| \leq M \) for all \( x \in \mathbb{R} \). Then if \( a < b \), then \( \frac{f(b) - f(a)}{b-a} = f'(x_0) \leq M \) for some \( x_0 \in (a, b) \). So \( |f(b) - f(a)| \leq M|b-a| \). Therefore, given \( \varepsilon > 0 \), if \( \delta < \varepsilon/M \) then \( |b-a| < \delta \) implies \( |f(b) - f(a)| < \varepsilon \). So \( f \) is uniformly continuous.

(d) Show that \( f(x) = \log(1 + x^2) \) is uniformly continuous on \( \mathbb{R} \). [TURN OVER.]

In view of the previous problem, it suffices to show that the derivative of \( f \) is bounded. So it suffices to show that \( \frac{2|x|}{1 + x^2} \) is bounded. If \( |x| \geq 1 \) then \( \frac{2|x|}{1 + x^2} \leq 2 \) while if \( |x| \leq 1 \) then also \( \frac{2|x|}{1 + x^2} \leq 2|x| \leq 2 \).

(4) (20 marks.) Recall that for \( x > 0 \) and \( a \in \mathbb{R} \), we define \( x^a = \exp(a \log(x)) \).

(a) Let \( a \in \mathbb{R} \). Show that \( \frac{d}{dx}(x^a) = ax^{a-1} \).

We use the chain rule to differentiate \( e^{a \log(x)} \). This gives \( \frac{a}{x} e^{a \log(x)} = ae^{-\log(x)} e^{a \log(x)} = ae^{(a-1) \log(x)} \).

(b) Let \( a > 1 \). Show that

\[
\int_1^N \frac{1}{x^a} \, dx = \frac{1}{1-a} (N^{1-a} - 1).
\]

By the previous problem, the derivative of \( \frac{1}{1-a} x^{1-a} \) is \( x^{-a} \). Therefore, by the fundamental theorem of calculus, we have

\[
\int_1^N \frac{1}{x^a} \, dx = \frac{1}{1-a} x^{-a} \bigg|_1^N.
\]

(c) Let \( I \) be a closed interval. Explain what is meant by the upper and lower Riemann sums \( S^+(f, P) \) and \( S^-(f, P) \) of a continuous function \( f : I \to \mathbb{R} \) with respect to a partition \( P \) of \( I \).

Let \( P = \{ x_0 < x_1 < \cdots < x_n \} \) be a partition. The upper Riemann sum \( S^+(f, P) \) is the sum

\[
\sum_{i=1}^n \sup_{x \in [x_{i-1}, x_i]} f(x)(x_i - x_{i-1}).
\]

The lower Riemann sum \( S^-(f, P) \) is the sum

\[
\sum_{i=1}^n \inf_{x \in [x_{i-1}, x_i]} f(x)(x_i - x_{i-1}).
\]
(d) For \( N \geq 2 \) and \( a > 1 \), show that
\[
\sum_{n=2}^{N} \frac{1}{n^a} \leq \int_{1}^{N} \frac{1}{x^a} dx.
\]

The function \( f(x) = x^{-a} \) is decreasing, because its derivative is \(-ax^{a-1}\), which is \(-a\) times the exponential of something, which must be negative. So the sum on the left hand side is the lower Riemann sum for the function \( f \) on \([1, N]\) with respect to the partition \( P = \{1, 2, \ldots, N\} \). The lower Riemann sum is \( \leq \) the integral since the integral is the supremum of the set of lower Riemann sums of all partitions \( P \).

(e) Show that if \( a > 1 \), then the series \( \sum_{n=1}^{\infty} \frac{1}{n^a} \) converges.

The \( N^{th} \) partial sum of the series is bounded above by the integral, whose value is
\[
\int_{1}^{N} \frac{1}{x^a} dx = \frac{1}{1-a} (N^{1-a} - 1).
\]

Since \( a > 1 \), the sequence
\[
b_N = \frac{1}{1-a} (N^{1-a} - 1)
\]
converges, and so is bounded. Therefore, the sequence of partial sums of \( \sum_{n=1}^{\infty} \frac{1}{n^a} \) is an increasing bounded sequence, so it converges.

(5) \( (20\) marks.) The following problem is set in an analysis exam which you are grading:

Problem: (10 marks) Suppose \( f : A \to \mathbb{R} \) where \( A \subset \mathbb{R} \). Let \( x \) be a cluster point of \( A \). Suppose \( \lim_{x \to a} f(x) = L \neq 0 \). Show that \( \lim_{x \to a} \frac{1}{f(x)} = \frac{1}{L} \).

A student writes the following solution:

"My solution:
\[
\left| \frac{1}{f(x)} - \frac{1}{L} \right| = \left| \frac{L-f(x)}{f(x)L} \right| = \frac{|f(x) - L|}{|f(x)| |L|} < \frac{\varepsilon}{|f(x)| |L|}
\]
if \(|f(x) - L| < \varepsilon\).

So given \( \varepsilon > 0 \), choose \( \delta > 0 \) such that, if \(|x - a| < \delta\), then \(|f(x) - L| < \varepsilon \cdot \inf |f(x)| \cdot |L|\). Then
\[
|x - a| < \delta \implies \frac{1}{f(x)} - \frac{1}{L} < \varepsilon.
\]
QED.”

(a) Comment on any aspects of the solution which you think are incorrect, or which could be improved.

*The proof is OK except for* $|f(x) - L| < \varepsilon \cdot \inf |f(x)| \cdot |L|$. *There is not necessarily any such thing as* $\inf |f(x)|$. *No set is specified over which the infimum is taken. The student should have shown that* $f$ *is bounded below near* $a$. *That is, there exists* $\delta_1$ *such that* $|x - a| < \delta_1$ *implies* $||f(x)| - |L|| \leq |f(x) - L| < |L|/2$ *and then* $|f(x)| \geq |L| - |L|/2 = |L|/2$. *Now replace the inf by* $|L|/2$ *in the above proof, and replace* $\delta$ *by the minimum of* $\delta$ *and* $\delta_1$. *Then the proof works.*

(b) How many marks (out of a maximum possible 10) would you award the student? Explain your answer.

*The proof is mostly, but not wholly, correct. Therefore, any answer* $< 10$ *is acceptable here.*

[END.]