MATH 2220 HW1 SOLUTIONS.

Homework 1. Due Wednesday 3 September.

(1) Section 1.3, p. 61–65

(a) # 8.

The volume is given by absolute value of the determinant

\[
\begin{vmatrix}
1 & 0 & 0 \\
0 & 3 & -1 \\
4 & 2 & -1
\end{vmatrix} = \begin{vmatrix}
3 & -1 \\
2 & -1
\end{vmatrix} = -1.
\]

So the volume is \(|-1|=1\).

(b) # 24.

The direction vector of the line is \((3, -2, 4)\), so the desired plane is the plane through \((2, -1, 3)\) orthogonal to \((3, -2, 4)\). This plane is given by the equation

\[3(x - 2) - 2(y + 1) + 4(z - 3) = 0.\]

(c) # 36.

Following the hint:

\[
\begin{vmatrix}
 a_1 - x & a_2 - y & a_3 - z \\
b_1 - x & b_2 - y & b_3 - z \\
c_1 - x & c_2 - y & c_3 - z
\end{vmatrix} = (a_1 - x, a_2 - y, a_3 - z) \cdot ((b_1 - x, b_2 - y, b_3 - z) \times (c_1 - x, c_2 - y, c_3 - z))
\]

In the notation of the problem this is \((A - P) \cdot ((B - P) \times (C - P))\). The problem asks us to show that \(P\) lies on the plane determined by \(A, B\) and \(C\) if and only if \((A - P) \cdot ((B - P) \times (C - P)) = 0\). (Note that this doesn’t make sense if \(A, B, C\) are collinear, so we must assume that they are not.) One way to see the result geometrically is that the vectors \(A - P, B - P\) and \(C - P\) all lie in a plane if and only if the volume of the parallelepiped determined by them is zero, and this volume is precisely \(|(A - P) \cdot ((B - P) \times (C - P))|\).
Another way to do the problem is to observe that

\[
D = \begin{vmatrix}
  a_1 - x & a_2 - y & a_3 - z \\
  b_1 - x & b_2 - y & b_3 - z \\
  c_1 - x & c_2 - y & c_3 - z \\
\end{vmatrix} = \begin{vmatrix}
  a_1 - x & a_2 - y & a_3 - z \\
  b_1 - a_1 & b_2 - a_2 & b_3 - a_3 \\
  c_1 - a_1 & c_2 - a_2 & c_3 - a_3 \\
\end{vmatrix}
\]

by subtracting the first row of the determinant from the other two rows. This is
the equation of a plane (to see this, expand the determinant along the top row).
Since the determinant \(D\) has value zero if \((x, y, z)\) is set to one of \(A, B\) or \(C\),
the plane determined by \(D = 0\) contains \(A, B\) and \(C\). Again, assuming that \(A, B\)
and \(C\) are not collinear, it is clear geometrically that they determine a unique
plane.

(2) Find a unit vector parallel to the line of intersection of the planes \(x - 2y + 5z = 2\)
and \(3x - y + 5z = 3\).

A vector orthogonal to the first plane is \((1, -2, 5)\) and a vector orthogonal to the
second plane is \((3, -1, 5)\) (read off the coefficients). A vector pointing along the line
of intersection of the planes must be orthogonal to both of these, so we can find one
by taking the cross product:

\[
\begin{vmatrix}
  i & j & k \\
  1 & -2 & 5 \\
  3 & -1 & 5 \\
\end{vmatrix}
\]

which equals \(-5i + 10j + 5k\). We want a unit vector, so we must divide by the norm,
which is \(\sqrt{5^2 + 10^2 + 5^2} = \sqrt{150}\). The desired vector is therefore

\[
\frac{1}{\sqrt{150}}(-5i + 10j + 5k).
\]

(3) Given the points \(P = (1, 2, 3), Q = (3, 5, 2)\) and \(R = (2, 2, 3)\) find:

(a) The area of the triangle \(PQR\).

Subtracting \(P\) from everything does not change the area of the triangle (it just
moves everything by \(-P\)). So we want the area of the triangle with vertices 0,
\(Q - P\) and \(R - P\). By the lectures, we know that the area of the parallelogram
determined by \(Q - P\) and \(R - P\) is \(\|(Q - P) \times (R - P)\|\), so the desired area is
half of this.

\[(Q - P) \times (R - P) = \begin{vmatrix} i & j & k \\ 2 & 3 & -1 \\ 1 & 0 & 0 \end{vmatrix} = -j - 3k\]

So \(\frac{1}{2} \| (Q - P) \times (R - P) \| = \frac{1}{2} \sqrt{1^2 + 3^2} = \sqrt{10}/2\).

(b) The distance from R to the line through P and Q.

There are several ways of doing it; here is one. First, we must find the line through P and Q. This line is given by \(P + t(Q - P), t \in \mathbb{R}\). Therefore, it is \((1, 2, 3) + t(2, 3, -1), t \in \mathbb{R}\). We need to find the point on this line which is closest to R. The distance from a general point on the line to R is \(\| (1 + 2t, 2 + 3t, 3 - t) - (2, 2, 3) \| = \|(-1 + 2t, 3t, -t)\| = \sqrt{(-1 + 2t)^2 + (3t)^2 + t^2}\). This equals \(\sqrt{1 - 4t + 14t^2}\). This will be minimized by the value of \(t\) which minimizes \(1 - 4t + 14t^2\), which we can find using calculus. The desired value of \(t\) satisfies \(-4 + 28t = 0\), so \(t = 4/28 = 1/7\). (The second derivative test guarantees that this is indeed a minimum, as it should be by geometric intuition.) The desired distance is therefore \(\sqrt{1 - 4/7 + 14/49}\).

An even faster method is the following. Let the desired distance be \(h\). Then the area of the triangle PQR is \(\frac{1}{2}h \cdot \text{dist}(P, Q)\), by the formula for the area of a triangle (half times base times height). So \(h = 2 \cdot \frac{\sqrt{10}}{2 \| (2,3,-1) \|} = \frac{\sqrt{10}}{\sqrt{14}}\).