

**Diameters of
Cayley graphs of $SL_n(\mathbb{Z}/k\mathbb{Z})$**

Work in collaboration with Martin Kassabov

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Tim Riley

$$\mathcal{A} := \begin{pmatrix} 1 & 1 & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \cdots & \\ & & & & 1 \end{pmatrix} \quad \text{and} \quad \mathcal{B} := \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & \cdots & \\ & & & \cdots & 1 \\ (-1)^{n-1} & & & & 0 \end{pmatrix}$$

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Theorem. *For all $k \geq 2$ and $n \geq 3$,*

$$\mathrm{Diam} \, \mathrm{Cay}(\mathrm{SL}_n(\mathbb{Z}/k\mathbb{Z}), \{\mathcal{A}, \mathcal{B}\}) \leq 3600 n^2 \ln k.$$

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is not an expander family.

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is not an expander family.

- It is a better result than you get using Property (T) and is constructive and efficient.
- The result is best possible because

$$|SL_n(\mathbb{Z}/k\mathbb{Z})| \sim k^{n^2-1}.$$

Broader Context

Fix $n \geq 3$.

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Problem. Give an elementary proof of the result of Gromov that $\mathrm{SL}_3(\mathbb{Z})$ admits an exponential Dehn function.

Problem. Prove Thurston's assertion that $\mathrm{SL}_n(\mathbb{Z})$ admits a quadratic isoperimetric function for all $n \geq 4$.

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For $i \neq j$ define e_{ij} to be the matrix with entry ij and all diagonal entries 1 and all other entries 0.

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I) Using Gaussian elimination, matrices in $SL_n(\mathbb{Z}/k\mathbb{Z})$ can be written as a product of n row matrices, n column matrices and n elementary matrices.

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Our proof of the theorem has three parts:

- I) Using Gaussian elimination, matrices in $SL_n(\mathbb{Z}/k\mathbb{Z})$ can be written as a product of n row matrices, n column matrices and n elementary matrices.
- IIa) Bit-row and bit-column matrices in $SL_n(\mathbb{Z})$ can be represented by words on \mathcal{A} and \mathcal{B} of length $< 50n$.
- IIb) Row and column matrices in $SL_n(\mathbb{Z})$ with entries in $\{-K + 1, \dots, K - 2, K - 1\}$ can be represented by words on \mathcal{A} and \mathcal{B} of length $\leq 1200n \ln K$.

IIa) Generating bit-row and bit-column matrices

For $m_3, \dots, m_n \in \{0, \pm 1\}$, matrices

$$\begin{pmatrix} 1 & 0 & m_3 & m_4 & m_5 & \cdots & m_n \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & 1 & & & \\ & & & & 1 & & \\ & & & & & \cdots & \\ & & & & & & 1 \end{pmatrix}$$

can be expressed as $N_1 e_{12} N_1^{-1} N_2 e_{12}^{-1} N_2^{-1}$ where N_1 and N_2 have the form

$$\begin{pmatrix} 1 & 0 & 0 & & & & \\ & 1 & -s_2 & -t_2 & & & \\ & & 1 & -s_3 & -t_3 & & \\ & & & 1 & -s_4 & \cdots & \\ & & & & 1 & \cdots & -t_{n-2} \\ & & & & & \cdots & -s_{n-1} \\ & & & & & & 1 \end{pmatrix}$$

for some $s_2, \dots, s_{n-1} \in \{0, \pm 1, \pm 2\}$ and $t_2, \dots, t_{n-2} \in \{0, 1\}$.

Such N_1 and N_2 can be expressed as

$$\begin{aligned}
 w &= \mathcal{B}^{-(n-2)} Q_{n-1} \mathcal{B} Q_{n-2} \mathcal{B} \dots Q_2 \mathcal{B} \\
 &= (\mathcal{B}^{-(n-2)} Q_{n-1} \mathcal{B}^{n-2}) (\mathcal{B}^{-(n-3)} Q_{n-2} \mathcal{B}^{n-3}) \dots (\mathcal{B}^{-1} Q_2 \mathcal{B}),
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where

$$Q_i := \mathcal{A}^{-s_i} [\mathcal{A}, \mathcal{B}^{-1} \mathcal{A} \mathcal{B}]^{-t_i} = \begin{pmatrix} 1 & -s_i & -t_i & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

and w has length $\leq 12n - 24$ as a word on \mathcal{A} and \mathcal{B} .

Then, by fixing up the second entry in row 1 and conjugating by a power of \mathcal{B} , we can produce any given bit-row matrix.

A similar proof works for bit-column matrices. Deduce –

Bit-row and bit-column matrices in $SL_n(\mathbb{Z})$ can be represented by words on \mathcal{A} and \mathcal{B} of length $< 50n$.

IIb) Generating row and column matrices

Fibonacci numbers:

$$F_0=0, \quad F_1 = 1, \quad F_{i+2}=F_{i+1} + F_i.$$

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Claim. For non-negative integers l ,

$$e_{12}^2(e_{21}e_{12})^l e_{13}(e_{21}e_{12})^{-l} e_{12}^{-1}(e_{21}e_{12})^l e_{13}^{-1}(e_{21}e_{12})^{-l} e_{12}^{-1},$$

equals $e_{13}^{F_{2l}}$ in $\mathrm{SL}_3(\mathbb{Z})$.

$$\begin{pmatrix} 1 & 0 & \mathbf{0} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{e_{12}^2} \begin{pmatrix} 1 & 2 & \mathbf{0} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{(e_{21}e_{12})^l} \begin{pmatrix} F_{2l+2} & F_{2l+3} & \mathbf{0} \\ F_{2l} & F_{2l+1} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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For arbitrary integers $K > 0$ we can write e_{13}^K as a word of length $\leq \ln K$ on $e_{12}, e_{21}, e_{23}, e_{13}$ by using –

Zeckendorf's Theorem.

Every positive integer K can be expressed as

$$K = F_{k_1} + F_{k_2} + \cdots + F_{k_r},$$

with $k_1 \geq 2$ and $k_{j+1} - k_j \geq 2$ for all $1 \leq j < r$.

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Replacing the e_{13} and e_{23} by bit-row matrices we can build sums of Fibonacci numbers in entries $3, 4, \dots, n$ of row 1, and hence produce arbitrary row matrices.

Produce arbitrary column matrices similarly.

These methods also yield –

- An upper bound of $O(\log k)$ on the *worst–case non–deterministic* complexity of the subtractive Euclid’s algorithm for computing the **gcd** of $n \geq 3$ integers (a_1, \dots, a_n) , where $k := \max \{|a_1|, \dots, |a_n|\}$.

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- An elementary, constructive proof of the Mozes–Lubotzky–Raghu-nathan Theorem for $\mathbf{SL}_n(\mathbb{Z})$ when $n \geq 3$.
- A *normal form* of *linearly bounded length* for $\mathbf{SL}_n(\mathbb{Z})$.