4. The forward implications are false. For a counterexample to the first, consider the (connected) union of these 2 closed unit discs in the plane,

\[ \{(a, b) \mid (a + 1)^2 + b^2 \leq 1\} \cup \{(a, b) \mid (a - 1)^2 + b^2 \leq 1\} \]

Which are tangent at the origin. This subset of \( \mathbb{R}^2 \) has an interior with 2 connected components.

For a counterexample to the second, consider the (connected) closed interval

\[ [0, 1] \subset \mathbb{R} \]

Its boundary has two connected components.

As for the converse, we can construct spaces without interior (so that \( \text{Int} A \) is vacuously connected) whose boundary is connected. Take \( \mathbb{Q} \subset \mathbb{R} \), for example. Then \( \text{Int} A = \emptyset \), and \( \partial A = \mathbb{R} \), so the converse is false.

5. (a) Consider the map

\[ \phi : \mathbb{R}^{n^2} \to \mathbb{R}^{n^2} \]

which takes a matrix \( A \) and forms the matrix \( \phi(A) = \{c_i \cdot c_j\}_{i,j} = A^T \cdot A \) whose \( i, j \) entry is the dot product of the \( i \)th and \( j \)th columns of \( A \). You might notice that the image of \( \phi \) is in the set of symmetric matrices (i.e., those such that \( A = A^T \), or equivalently, \( a_{i,j} = a_{j,i} \) for each \( i, j \) of
any matrix in the image), since dot product is symmetric. Also notice that

\[ \phi|_{O(n)} \equiv I, \]

that is, \( \phi \) restricted to \( O(n) \) is identically the \( n \times n \) identity matrix.

Conversely, suppose \( \phi(B) = I \). Then the columns of \( B \) must all be unit length (hence the 1’s on the diagonal) and orthogonal (hence the zeros off the diagonal), so \( B \in O(n) \).

Therefore, \( \phi^{-1}(I) = O(n) \).

We must argue that \( \phi \) is continuous, but this easily follows from the fact that \( \phi \) is polynomial, i.e., each component of \( \phi(A) \) is a sum of products of the coordinates of \( A \), so that by varying \( A \) we are varying by a polynomial in each coordinate of \( \phi(A) \).

Finally, the inverse image of a closed set under a continuous map is closed, so \( O(n) \) is closed.

(b) By a theorem in Hatcher (in fact the Heine-Borel theorem), if we show that \( O(n) \) is bounded, we get, with part (a), that \( O(n) \) is compact. But here simply note that no column of \( A \in O(n) \) can contain a coordinate \( c \) with

\[ |c| > 1. \]
And hence, $O(n)$ is contained in the cube

$$\{ A \in \mathbb{R}^{n^2} \mid |a_{i,j}| \leq 1, \text{ for all } i, j \}.$$