4. One direction is trivial, since if \( f \) is continuous, and \( B \) is a basis set in \( Y \), then \( B \) is open, so \( f^{-1}(B) \) is open in \( X \). Conversely, suppose \( f^{-1}(B) \) is open in \( X \) for every \( B \in \mathcal{B} \). For \( U \) an arbitrary open subset of \( Y \) we have some

\[
\{B_\alpha\}_\alpha \subset \mathcal{B}
\]

so that

\[
U = \bigcup_\alpha B_\alpha
\]

and hence

\[
f^{-1}U = f^{-1}\bigcup_\alpha B_\alpha = \bigcup_\alpha f^{-1}B_\alpha,
\]

because inverse image of a map commutes with union. Therefore, \( f^{-1}(U) \) is the union of open sets, and so is open. This shows \( f \) is continuous.

5. (a) By 4. it is enough to show the inverse image of basis sets are open. Let \((a, b) \subset \mathbb{R}\) be an open interval. Then

\[
f^{-1}((a, b)) = \{(x, y) \in \mathbb{R}^2 \mid a < x + y < b\}.
\]

Consider the topology on \( \mathbb{R}^2 \) generated by sets

\[
\{(x, y) \mid |x - x_0| + |y - y_0| < \epsilon \mid \epsilon > 0, \ (x_0, y_0) \in \mathbb{R}^2\}.
\]

By problem 2 on homework set 2 this generates the usual topology. For a point \((x_0, y_0)\) so that \( a < x_0 + y_0 < b \), let

\[
\epsilon = \min\{x_0 + y_0 - a, b - (x_0 + y_0)\}.
\]
Then the ball in this metric (recall it looks like a diamond) centered at 
\((x_0, y_0)\), of radius \(\epsilon\) is contained in \(f^{-1}((a, b))\), so \(f\) is continuous.

(b) Similarly to (a), take some \((x_0, y_0)\) in \(\mathbb{R}^2\) so that \(x_0y_0 \in (a, b) \subset \mathbb{R}\).

Suppose, first, that \(x_0, y_0\) is off the axis, so neither coordinate is 0.
Let \(B_\epsilon\) be an open square, centered at \((x_0, y_0)\), with side lengths of \(2\epsilon\),
chosen small enough to be contained in the quadrant that \((x_0, y_0)\) is in.
If \((x_0, y_0)\) is in the first quadrant, then the image of \(B_\epsilon\) under \(f\) is

\[
(x_0y_0 - (x_0 + y_0)\epsilon + \epsilon^2, x_0y_0 + 2\epsilon + \epsilon^2).
\]

Clearly \(\epsilon\) can be chosen small enough so that this interval is contained
in \((a, b)\), hence this square is contained in \(f^{-1}((a, b))\), and since the
square is a standard product basis set, and \(\mathbb{R}^2\) has the topology of a
product, such a square is open, so that \(f\) is continuous.

If \((x_0, y_0)\) is in the second quadrant, then the image of \(B_\epsilon\) under \(f\) is

\[
(x_0y_0 + \epsilon y - x\epsilon - \epsilon^2, x_0y_0 - \epsilon y + x\epsilon - \epsilon^2),
\]

and the proof goes through in the same way.

If \((x_0, y_0)\) is on a coordinate axis, then \(\epsilon\) can be chosen so that \(B_\epsilon\) is
arbitrarily close to 0, (i.e., contained in an arbitrarily small neighborhood around 0), in exactly the same way, and the proof goes through
as above.