Topology Math 4530  
Solutions to set 12

1. (a) We will apply the Seifert-Van Kampen theorem to each. For the first topological space, let $U$ be a neighborhood of the boundary of the triangle, which deformation retracts to it, and let $V$ be the interior disk, so that $U \cap V \cong S^1$. Now, $U, V, U \cap V$ are all path connected, and $V$ is homotopically trivial. $U$ is easily seen to deformation retract to a circle. When we inject the generator of $U \cap V$ into $U$ we get the word $aaa^{-1}a = a$, so by the Seifert-Van Kampen theorem the first topological space has trivial fundamental group.

(b) Identical solution as done in part (a) except now, as a relation in $U \cup V$ we get the word $a^3$, so that the fundamental group of this space is $Z/3Z \cong C_3$.

2. (a) The non-orientable surface with genus $g$ is given as a sphere with $g$ discs removed and each sewn together with cross-caps. We image each of these cross-caps on the surface of a sphere, and move them all towards each other at one point, so that one has what looks like a wedge (or bouquet) of cross-caps, all joined together at a single vertex. It should be clear that this space is homotopically equivalent to $\Psi_g$, if not homeomorphic. Then cutting along the cross-caps, from the central vertex, one obtains the $2n$-gon with side identifications $a_1a_1a_2a_2 \cdots a_ga_g$. From this $2n$-gon we calculate the fundamental group by an similar procedure as in (1), by using the Seifert-Van Kampen theorem, and obtain $a_1a_1a_2a_2 \cdots a_ga_g$ as the single relation of this group on $g$ generators.

(b) Abelianizing this group gives an abelian group on $g$ generators. Note that, picking the $g$th generator (arbitrarily), we can express any word in this group uniquely as one which has $a_g$ to a power either 1 or 0, since we can increase or decrease the exponents of all generators by 2, by multiplying by $a_1^2 \cdots a_g^2$ a number of times, or by its inverse. Note also that any integer powers are possible for the other exponents, so that this expression gives us one of any elements in $Z^{g-1} \times C_2$. It is trivially verified that this map is an isomorphism.
3. (a) Applying the Wirtinger construction to the unlink gives the free group on 2 generators, $F_2$. For the Hopf link we have 2 relations $aba^{-1}b^{-1}$, $bab^{-1}a^{-1}$, and each of these confirms that this group is $\mathbb{Z}^2$, the free abelian group on 2 generators.

(b) For the figure on the left we will label the generators, which will be strands from one undercrossing to the next, with the symbols $a_i$, beginning with the top center strand, and following in the order obtained by moving to the right. Hence the right most strand is $a_2$, the bottom center is $a_3$, etc. We will label the crossings with the symbols $C_i$, beginning with the crossing where $a_1$ crosses over $a_2$, and proceeding by a clockwise direction, moving down, then left and back up. The 6 crossings give us 6 relations. These are

$$r_1 = a_1a_2a_1^{-1}a_3^{-1},$$
$$r_2 = a_3a_1a_3^{-1}a_2^{-1},$$
$$r_3 = a_2a_3a_2^{-1}a_4^{-1},$$
$$r_4 = a_6a_4a_6^{-1}a_5^{-1},$$
$$r_5 = a_5a_6a_5^{-1}a_1^{-1},$$
$$r_6 = a_1a_5a_1^{-1}a_6^{-1}. $$

The analogous procedure for the knot on the right (with an analogous labeling) gives relations

$$s_1 = a_3a_1a_3^{-1}a_2^{-1},$$
$$s_2 = a_2a_3a_2^{-1}a_4^{-1},$$
$$s_3 = a_4a_2a_4^{-1}a_3^{-1},$$
$$s_4 = a_6a_4a_6^{-1}a_5^{-1},$$
$$s_5 = a_5a_6a_5^{-1}a_1^{-1},$$
$$s_6 = a_1a_5a_1^{-1}a_6^{-1}. $$

(c) The final 3 relations in the each list above are identical to their corresponding relation in the other list. The product $a_6^{-1}r_6r_5r_4a_6 = 1$ gives the relation $a_1a_4^{-1}$ so that $a_1 = a_4$. We can use the relations to write either group solely in terms of 3 generators, say $a_1, a_2, a_6$, from which point it is clear that the two groups are isomorphic.
4. (a) The cross-cap like seam down the middle can be seen to reverse orientation, when one traverses through it, then around it on the inside. Therefore the surface we are looking at is non-orientable. By the classification of surfaces, we can determine the surface by calculating its Euler characteristic. By placing vertices at the corners of the seam, we can decompose the space, from left to right, as follows. One 2-cell; one vertex and two 1-cells; two 2-cells and two 1-cells (a figure eight cartesian product with an open interval); one vertex and two 1-cells; one 2-cell. Taking the alternating sum over the dimensions of these gives 2-6+4=0, so the surface is a Klein bottle. These result can also be obtained by a cutting and glueing argument, similar to that done in homework 7.

(b) We give the space a cellular decomposition, as in (a). Here, we can “snip” the flat mobiüs handles, by putting edges across them, connected to the boundary edges by a total of 4 vertices. Computing the Euler characteristic gives the alternating sum 4 − 6 + 1 = −1, so that compactifying the space with the addition of a 2-cell, one must obtain a Klein bottle. This result can also be obtained by a cutting and glueing argument similar to that done in homework 7.

5. As was done on homework 10, we will visualize the genus $g$ orientable surface as being centered in $\mathbb{R}^3$ in a nice way. If $g = 2k + 1$ is odd, then picture $\Sigma_g$ as a torus, with $2k$ handles, $k$ of which extend, like a chain, in one direction, while the other $k$ extend in the antipodal direction. If $g = 2k$ is even then replace the torus in the middle with a sphere, still with $2k$ handles. Quotienting by the antipodal map turns the first surface ($g = 2k + 1$) into the non-orientable surface of genus $2k + 2$, as it is now a Klein bottle with $k$ handles, each of which contribute 2 non-orientable genera. In the second case, the center surface becomes a projective plane, and the resulting surface is genus $2k + 1$. By the classification theorem, this accounts for all non-orientable surfaces.

6. (a) Note that there is a continuous surjective map from $S^1 \subset \mathbb{C}$ to $X$. For example, send $1, -1$ to $b, d$ respectively, and send the open northern and southern hemispheres to $a, c$ respectively. It is easily verified
that this is a continuous map. $S^1$ is path connected, and therefore $X$ is path connected.

(b) We will find a loop in $X$ which is not nullhomotopic. Let $f$ be the function described in (a), and let $\gamma$ be the standard generating loop based at 1 in $S^1$, given by $\gamma(t) = e^{it}$. Assume that there is a nullhomotopy of $f \circ \gamma$. Then we have a continuous map $F$ of a disk $D^2$ into $X$, such that the boundary, $S^1$, is sent via $f$ into the loop $f \circ \gamma$. The preimage of this disk is the disjoint union of 4 regions, corresponding to where they are sent in $X$. The preimages of $a$ and $c$ are disjoint and the boundary of $F^{-1}(\{a\})$ must be contained in $F^{-1}(\{b, d\})$. Since the northern hemisphere of $\partial D^2$ is mapped to $a$, and the points 1, $-1$ to $b, d$, respectively, then there must be a path $p$ contained in $F^{-1}(\{b, d\})$ which connects $-1$ to 1, but this path is the disjoint union of the two closed sets $F^{-1}(\{b\}) \cap p(I), F^{-1}(\{d\}) \cap p(I)$ (here $I$ is the interval which is a domain for $p$). This is a contradiction, as this path is connected, so it cannot be that $f \circ \gamma$ is contractible.