Math 4530 — Topology. Homework 8
Due in class on 27th October, 2009.
Please declare any collaborations with classmates; if you find solutions in books or on-line, acknowledge your sources — in either case, write your answers in your own words.
Please attempt all questions and justify your answers.

1. [For class discussion]
(a) Show that the identity element $e$ of a group $G$ is unique in the sense that if $f \in G$ also satisfies
\[ fg = gf = g \] for all $g \in G$, then $e = f$.
(b) Suppose $G$ is a group for which $x^2 = e$ for all $x \in G$. Show that $G$ is abelian (that is, $ab = ba$ for all $a, b \in G$).

2. [For class discussion] A group $G$ is a topological group when it is equipped with a topology such that the functions $G \times G \to G$, defined by $(x, y) \mapsto xy$, and $G \to G$, defined by $x \mapsto x^{-1}$, are continuous. Prove that the connected component $C$ of a topological group $G$ containing the identity $e$ has the property that $x^{-1}$ and $xy$ are in $C$ for every $x, y \in C$. (So $C$ is what is known as a subgroup of $G$.)

3. [For class discussion] Which of the following are groups? You are encouraged to keep your explanations brief.
(a) The four complex numbers $1, i, -1, -i$, under multiplication.
(b) The natural numbers $\mathbb{N}$ under addition.
(c) The integers $\mathbb{Z}$ under addition.
(d) The integers $\mathbb{Z}$ under multiplication.
(e) The rationals $\mathbb{Q}$ under multiplication.
(f) The integers modulo 12 under addition.
(g) The non–zero integers modulo 6 under multiplication.
(h) The non–zero integers modulo 11 under multiplication.
(i) The bijections $S \to S$ under composition, where $S$ is a non–empty set.
(j) The isometries of a metric space $X$, under composition — that is, the bijections $f : X \to X$ such that $d(a, b) = d(f(a), f(b))$ for all $a, b \in X$.
(k) The $n \times n$ matrices with integer valued entries whose determinants are non–zero, under multiplication.
(l) The $n \times n$ matrices with real valued entries whose column vectors comprise an orthonormal basis for $\mathbb{R}^n$, under multiplication.
(m) The continuous surjections $X \to X$ under composition, where $X$ is a topological space.
(n) The homeomorphisms $X \to X$ under composition, where $X$ is a topological space.

4. [Munkres] Let $X$ be the figure–eight space — that is, the subspace of the plane consisting of the circles of radius 1 centered at $(0, 1)$ and $(0, -1)$. Let $Y$ be theta space — that is, the subspace of the plane consisting of $S^1$ together with the diameter from $(-1, 0)$ to $(1, 0)$. Describe maps $f : X \to Y$ and $g : Y \to X$ that are homotopy inverses of each other.

5. Suppose $X$ is a topological space with a base point $x$. Let $I$ denote the unit interval $[0, 1]$. Let $\gamma_0 : I \to X$ be the map $\gamma_0(s) = x$ for all $s \in I$. Suppose $\gamma : I \to X$ is continuous map with $\gamma(0) = \gamma(1) = x$. Define $\tilde{\gamma} : I \to X$ by $\tilde{\gamma}(s) = \gamma(1 - s)$. Give an explicit based homotopy $f_t : I \to X$ ($t \in I$) between $\tilde{\gamma} \ast \gamma$ and $\gamma_0$. Illustrate your homotopy with an appropriate sketch.

Proving this and the corresponding result for $\gamma \ast \tilde{\gamma}$ amounts to checking the existence-of-inverses axiom for a fundamental group.

Explanation of the terminology — Saying the homotopy $f_t$ is based means $t \mapsto f_t(0)$ and $t \mapsto f_t(1)$ are both constant functions. If $f, g : I \to X$ are continuous maps with $f(1) = g(0)$, then $f \ast g$ is defined to be the map $h : I \to X$ such that $h(s) = f(2s)$ for $s \in [0, 1/2]$ and $h(s) = g(2s - 1)$ for $s \in [1/2, 1]$. 

TRR