Math 4530 — Topology. Homework 7

Due in class on 20th October, 2009.

Please declare any collaborations with classmates; if you find solutions in books or on-line, acknowledge your sources — in either case, write your answers in your own words.

Please attempt all questions and justify your answers.

1. [For class discussion] Let $X$ be the quotient $\mathbb{R}/\sim$ where $r \sim s$ if and only if $r - s \in \mathbb{Z}$. Explain carefully why $X$ is homeomorphic to the circle $S^1$.

2. [For class discussion] The figure below depicts, from left to right, a sphere, a projective plane, a sphere with one cross-cap, a sphere with two cross-caps, and a Klein bottle.

Use (informal) cutting and gluing arguments to show that the sphere with one cross-cap is homeomorphic to the projective plane and that the sphere with two cross-caps is homeomorphic to the Klein bottle.

![Diagram of geometric shapes]

3. [Adapted from Viro et. al., Qu. 21.25] Identifying $\mathbb{R}^4$ with $\mathbb{C}^2$ allows us to express the 3-sphere $S^3 = \{ (x_1, x_2, x_3) \in \mathbb{R}^4 \mid \sum_{i=1}^{n} x_i^2 = 1 \}$ as $\{ (z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1 \}$.

By considering the three subspaces of $S^3$ determined by the equations $|z_1| = |z_2|$, $|z_1| \leq |z_2|$, and $|z_2| \leq |z_1|$, respectively, show that $S^3$ is homeomorphic to two copies of the solid torus $S^1 \times D^2$ with points on the boundary of one copy identified with points on the boundary of the other via the map $f : S^1 \times S^1 \to S^1 \times S^1$ given by $f(x, y) = (y, x)$.

TRR