Math 4530 — Topology. Homework 5

Due in class on 29nd September, 2009.

Please declare any collaborations with classmates; if you find solutions in books or on-line, acknowledge your sources — in either case, write your answers in your own words.

Please attempt all questions and justify your answers.

1. [For class discussion — from Hatcher.] Show that \( \mathbb{R} \) with the cofinite topology is compact.

2. [For class discussion — from Hatcher.] Show that if \( A \) and \( B \) are compact subspaces of a space \( X \), then so is \( A \cup B \). If, in addition, \( X \) is Hausdorff, show that \( A \cap B \) is compact.

3. [For class discussion — from Hatcher.] Show that if \( X \) is compact and \( f : X \to \mathbb{R} \) is continuous, then \( f \) is bounded and takes on a minimum and a maximum value.

4. [From Munkres.] If \( A \) is a connected subspace of \( X \), does it follow that \( \text{Int} A \) and \( \partial A \) are connected? Does the converse hold?

5. [From Hatcher.] Consider the set (known as the orthogonal group) \( O(n) \) consisting of all \( n \times n \) orthogonal matrices — that is, the \( n \times n \) matrices whose columns form an orthonormal basis \( v_1, \ldots, v_n \) for \( \mathbb{R}^n \). We put a topology on \( O(n) \) by regarding it as a subspace of \( \mathbb{R}^{n^2} \), taking the \( n^2 \) entries of a matrix in \( O(n) \) as the coordinates of a point in \( \mathbb{R}^{n^2} \).

   (a) Show that \( O(n) \) is a closed subset of \( \mathbb{R}^{n^2} \) by considering the dot products \( v_i \cdot v_j \) of the columns of matrices in \( O(n) \) as functions \( \mathbb{R}^{n^2} \to \mathbb{R} \).

   (b) Show that \( O(n) \) is compact.

TRR