Math 4530 — Topology. Homework 4

Due in class on 22nd September, 2009.

Please declare any collaborations with classmates; if you find solutions in books or on-line, acknowledge your sources — in either case, write your answers in your own words.

Please attempt all questions and justify your answers.

1. [For class discussion.] Prove that the following characterisations of connectedness of a topological space \( X \) are equivalent.
   (a) Whenever \( X \) is expressed as the disjoint union \( U \sqcup V \) of two open subsets, one of \( U \) and \( V \) is empty.
   (b) There is no continuous surjection from \( X \) to the two-point space \( \{0, 1\} \) with the discrete topology.

2. [For class discussion — from Hatcher.] From the fact that an interval \([a, b]\) is connected, deduce the Intermediate Value Theorem: if \( f: [a, b] \to \mathbb{R} \) is continuous and \( f(a) < c < f(b) \) then there exists \( x \in [a, b] \) with \( f(x) = c \).

3. [For class discussion.]
   (a) [From Hatcher.] Show that if a space \( X \) has only finitely many connected components, then these components are open subsets of \( X \).
   (b) Give an example to show this result can fail when \( X \) has infinitely many connected components.

4. [From Hatcher.] Show that the subspace \( X \subset \mathbb{R}^2 \) consisting of all points \((x, y)\) such that at least one of \( x \) and \( y \) is rational, is connected.

5. Sort the following topological spaces into equivalence classes under homeomorphism. Unless otherwise stated, \( \mathbb{R} \) and \( \mathbb{R}^2 \) have their usual topologies, as does the complex plane (identify \( \mathbb{C} \) with \( \mathbb{R}^2 \)). Please fully justify your answer. When writing down an explicit homeomorphism is awkward, an informal description will suffice.
   (A) \( \mathbb{R} \),
   (B) \( \mathbb{R} \) with the cofinite topology — that is, \( U \subset \mathbb{R} \) is open when either \( U = \emptyset \) or \( \mathbb{R} \setminus U \) is finite,
   (C) \( \mathbb{Q} \) with the subspace topology inherited from \( \mathbb{R} \),
   (D) \( \mathbb{Q} \) with the discrete topology,
   (E) \( \mathbb{Z} \) with the subspace topology inherited from \( \mathbb{R} \),
   (F) \( (0, 1) \) with the subspace topology inherited from \( \mathbb{R} \),
   (G) \( D^2 = \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1 \} \) with the subspace topology inherited from \( \mathbb{R}^2 \),
   (H) \( \{ (x, y) \in \mathbb{R}^2 \mid x^2 - y^2 = 1 \} \) with the subspace topology inherited from \( \mathbb{R}^2 \),
   (I) \( [0, 1]^2 \subset \mathbb{R}^2 \) with the subspace topology inherited from \( \mathbb{R}^2 \),
   (J) \( \{ (x, \sin \frac{1}{x}) \mid x > 0 \} \subset \mathbb{R}^2 \) with the subspace topology inherited from \( \mathbb{R}^2 \),
   (K) the letter \( Y \) — or, more formally, the subspace \( S_0 \cup S_{2\pi/3} \cup S_{4\pi/3} \) of the complex plane, where \( S_\theta := \{ re^{i\theta} \mid 0 \leq r < 1 \} \).