DIFFERENTIAL GEOMETRY, SYMPLECTIC TOPOLOGY, AND THE DARBOUX THEOREM

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In one sense, both differential geometry and symplectic topology resemble the usual sort topology with constraints imposed on the continuous functions. Indeed, the symplectomorphisms form a subset of the diffeomorphisms, which in turn form a subset of the homeomorphisms. In this regard, those who practice differential geometry or symplectic topology really practice topology “with their hands tied behind their back”. Paradoxically, the same ropes that bind may as well be extra limbs, for the additional structures imposed on smooth and symplectic manifolds really open doors to interesting topics in mathematics. My only hope for the ensuing bit of prose is that it might impart to the reader at least a taste of this dichotomy. I’ll start by giving an overview of the structures imposed on smooth and symplectic manifolds as well as the associated constraints placed on homeomorphisms in the presence of these structures. From there, I’ll give a qualitative account of two interesting results from symplectic topology, namely a theorem due to Mikhail Gromov and another due to Jean Darboux. Finally, I’ll give an outline of the latter theorem’s proof to illustrate a use of the machinery packaged with symplectic manifolds.

Smooth and Symplectic Manifolds (without boundary)

As symplectic manifolds are smooth manifolds, pedagogically, it’s best to consider smooth structure before symplectic structure. Broadly speaking, a smooth manifold consists of a topological manifold $M$ where the collection of charts serves as a crutch for performing calculus via the local identification of the manifold with open subsets of $\mathbb{R}^n$. To be more precise, I first need the definition of compatible charts $(U, \phi : U \to A)$ and $(V, \psi : V \to B)$ (note: $A, B \subseteq \mathbb{R}^n$). Two charts are compatible when the transition maps $f = \phi' \circ (\psi')^{-1}$ and $f^{-1} = \psi' \circ (\phi')^{-1}$, where $\phi'$ and $\psi'$ are the restrictions to $W$ of $\phi$ and $\psi$ respectively, are continuously differentiable on their domains. A maximal covering of $M$ by compatible charts comprises a maximal atlas $\mathcal{A}$. $\mathcal{A}$ together with $M$ then forms a smooth manifold. From here on, I’ll simply refer to the underlying space $M$ when I refer to a smooth manifold $(M, \mathcal{A})$.

This additional structure imposed on a topological manifold allows for a nice definition of a smooth function $F : M \to N$ between smooth manifolds $M$ and $N$. Let $(U, \phi)$ and $(V, \psi)$ be charts at $p \in M$ and $F(p) \in N$, respectively. $F$ is smooth at $p$ when $\psi \circ F' \circ \phi^{-1}$, where $F'$ is the $F$’s restriction to $U$, is continuously differentiable at $\phi(p)$. It comes as no surprise that $F$ is smooth when it is smooth at every point $p \in M$. Note, in particular, that $F$ must be continuous if smooth. Note also the crutch I mentioned earlier appearing in this definition; locally, $F$ can be considered a function between open subsets of $\mathbb{R}^n$ because of the homeomorphisms packaged in the charts at $p$ and $F(p)$. Finally, it’s important to
realize that the compatible nature of the charts covering $N$ and $M$ ensures that smoothness is independent of chart choice.

Smooth functions between smooth manifolds play nicely with the smooth manifold structure in the same way that continuous functions play nicely with topological structure because the composition of two smooth functions yields another smooth function. Furthermore, they provide the natural way to identify pairs of manifolds. $M$ and $N$ are diffeomorphic when there is a smooth bijection $F : M \to N$ with smooth inverse. The properties of diffeomorphisms not only allow for the identification of $M$’s points with $N$’s, but of $M$’s maximal atlas with with $N$’s maximal atlas. To see this, construct the bijective map $\hat{F} : A_M \to A_N$ such that

$$\hat{F}(U, \phi) = (F(U), \phi \circ F^{-1}).$$

That the image of a chart in $A_M$ forms a chart compatible with every element of $A_N$ follows directly from $F$’s diffeomorphism status. Bijectivity follows from $F$’s bijection status. Thus, all of the structure present on $M$ coincides with the structure on $N$ modulo diffeomorphism. In particular, $M$ is homeomorphic to $N$. But I emphatically stress that diffeomorphism is a restricted sort of homeomorphism. To see examples of homeomorphic spaces that fail to be diffeomorphic, see the Wikipedia article on smooth manifolds.

Beyond giving meaning to smooth functions, the structure on a smooth manifold makes it possible to construct algebraic images of the manifold via tangent spaces akin to the way the fundamental group provides algebraic images of topological spaces. Given a point $p \in M$, the tangent space at $p$, $T_pM$, is the set of linear derivations acting on functions $f : M \to \mathbb{R}$ smooth at $p$. Thus, if $X \in T_pM$, $f, g : M \to \mathbb{R}$ are smooth at $p$, and $a, b \in \mathbb{R}$, then

$$X(af + bg) = aXf + bXg \in \mathbb{R}$$

$$X(fg) = f(p)Xg + g(p)Xf.$$

It turns out that $T_pM$ is a vector space with dimension equal to the dimension of the $M$. So intuitively, the tangent space at a point on a manifold is the collection of directed arrows originating at the point $p$ that lay ”tangent” to the manifold (this can be made more precise, see [1]).

Again making the analogy with the fundamental group, tangent spaces give rise to algebraic images of smooth functions similar to the way the fundamental groups give rise to algebraic images of continuous functions. Suppose $F : M \to N$ is smooth and $p \in M$. $F_* : T_pM \to T_{F(p)}N$ is a linear map, the pushforward of $F$, between tangent spaces defined by

$$(F_*X)(f) = X(f \circ F),$$

where $f : N \to \mathbb{R}$ is smooth at $F(p)$. Not surprisingly, if $F$ is a diffeomorphism, then $F_*$ is bijective. Furthermore, this correspondence between smooth functions and linear maps between tangent spaces naturally leads to a correspondence between these functions and linear maps between the spaces dual to the tangent spaces, the cotangent spaces. The
pullback of $F$, $F^*: T^*_F(p)N \rightarrow T^*_pM$ is defined by

$$(F^*\alpha)(X) = \alpha(F_*X),$$

where $\alpha \in T^*_F(p)N$ and $X \in T_F(p)N$.

The pullback construction stands out because it generalizes to a correspondence between (differential) cotangent vector fields and (differential) covariant tensor fields on smooth manifolds $M$ and $N$. A cotangent vector field is a smooth assignment of a cotangent vector $\alpha_p \in T^*_pN$ to every point $p \in N$. The notation I'll use is if $\alpha$ is a cotangent vector field, then $\alpha_p$ is the cotangent vector assigned to $p \in N$. Thus the pullback of $F: M \rightarrow N$ sends cotangent vector fields on $N$ to cotangent vector fields on $M$ via the point-wise prescription

$$(F^*\alpha)_p(X) = \alpha_{F(p)}(F_*X),$$

where $\alpha$ is a cotangent vector field on $N$ and $X \in T_pM$. In a similar manner, the pullback sends covariant tensor fields of type $(0,s)$ (these are defined in a similar manner as cotangent vector fields) on $N$ to covariant tensor fields of type $(0,s)$ on $M$:

$$(F^*\omega)_p(X_1, ..., X_s) = \omega_{F(p)}(F_*X_1, ..., F_*X_2).$$

Because covariant tensor fields are defined on entire smooth manifolds, they can be used to impose additional constraints on diffeomorphisms. This is analogous to the way a maximal atlas allowed for placing constraints on homeomorphisms. Indeed, this is precisely where symplectomorphisms, the equivalences between symplectic manifolds, come into play. A symplectic manifold is a smooth manifold together with a closed, nondegenerate, 2-form field $\omega$. In this context, closed means that the exterior derivative of $\omega$ vanishes. I won’t go into a digression here on the exterior derivative because the closed property can just be considered an undefined constraint on $\omega$ for the purpose of this paper. However, it’s worth noting that a closed form is something of a generalization of a vector field with vanishing curl. Pressing on, a symplectomorphism between symplectic manifolds $(M,\omega_M)$ and $(N,\omega_N)$ is a diffeomorphism $F: M \rightarrow N$ that preserves the symplectic structure in the sense that the 2-form fields are preserved under pullback:

$$\omega_M = F^*\omega_N.$$
between symplectic manifolds that gives a symplectomorphism between it’s domain and image. The standard symplectic form on \( \mathbb{R}^{2n} \) is given by

\[
\omega = \sum_{i=1}^{n} x_i^* \wedge y_i^*
\]

where the \( x_i \) and \( y_i \) are the coordinate functions on \( \mathbb{R}^{2n} \) (note that I’m identifying \( \mathbb{R} \) with it’s tangent space). Intuitively, the theorem says that you can’t take an open ball with symplectic structure and “squeeze” it inside of a cylinder while respecting the symplectic structure. It’s not hard to imagine taking such a ball, ignoring the symplectic structure, and smoothly stretching it to fit inside such a cylinder. Thus it’s easy to find a smooth embedding no matter how \( r \) is related to \( R \) (of course don’t let \( r \) go to 0 while \( R \) is finite). The diffeomorphism associated with such a smooth embedding then gives a diffeomorphism that fails to be a symplectomorphism.

Gromov’s non-squeezing theorem plays a foundational role in the study of symplectic topology, the study of global properties of symplectic manifolds. But perhaps the oldest interesting result important to this field concerns local properties of symplectic manifolds. The result, known as the Darboux Theorem, states that all symplectic manifolds of a given dimension are locally symplectomorphic. Thus, an ant crawling along a symplectic manifold \( M \) would have the same subjective experience as another ant crawling along any other symplectic manifold \( N \). This suggests that the interesting properties of symplectic manifolds are more global in nature. The utility of this theorem in symplectic topology comes from the implication that the entire symplectic manifold can be covered with charts containing open sets that are symplectomorphic to open subsets of \( \mathbb{R}^{2n} \) with the standard symplectic form defined above. Such a covering by charts can considerably simplify calculations [4].

**Proof Outline: The Darboux Theorem [adapted from PlanetMath.org]**

Before proving the theorem, I should define what it means for symplectic manifolds \( M \) and \( N \) to be locally symplectomorphic. What’s required is that for each \( p \in M \) there is an open neighborhood of \( p, U \), and an open subset \( V \) of \( N \) such that \( U \) and \( V \) are symplectomorphic. Here I’m implicitly assuming \( U \) and \( V \) are given the structure of symplectic manifolds in the standard way for open subsets. Specifically, if \( U \) is an open subset of a symplectic manifold \( M \), turn it into a smooth manifold by restricting every coordinate chart on \( M, (V, \phi) \), to \( (U \cap V, \phi') \), where \( \phi' \) is the restriction of \( \phi \) to \( U \cap V \). Now turn this smooth manifold into a symplectic manifold by restricting the 2-form field on \( M \) to \( U \).

To begin the proof outline, note that to prove the Darboux theorem, it’s enough to show that every symplectic manifold \( M \) with symplectic form \( \omega' \) is locally symplectomorphic to \( \mathbb{R}^{2n} \) with it’s standard symplectic structure (defined perviously). So the first step in the proof is to choose a chart \( (U, \phi) \) at some point \( p \in M \) and use this chart to give \( \phi(U) \) symplectic structure. To see how this works, first note that \( \phi \) is actually a diffeomorphism when \( U \) and \( \phi(U) \) are given the usual smooth manifold structure. Thus \( \phi^* \) is bijective and \( \omega = \phi^* \omega' \) is a symplectic form on \( \phi(U) \); this makes \( \phi(U) \) into a symplectic manifold symplectomorphic to \( U \). If the theorem were true for open subsets of \( \mathbb{R}^{2n} \) (regarded as
smooth manifolds) paired with arbitrary symplectic forms, then there would be a neighborhood $W$ of $\phi(p)$ symplectomorphic to an open subset $W'$ of $\mathbb{R}^{2n}$ with its usual symplectic structure via a symplectomorphism $F : W \to W'$. Then, if $\phi'$ is $\phi$'s restriction to $\phi^{-1}(W)$, $F \circ \phi'$ would be a symplectomorphism (it’s a composition of symplectomorphisms!) and the Darboux theorem would be proven in general.

So let's assume $U \subseteq \mathbb{R}^{2n}$ is a symplectic manifold with symplectic form $\omega$ and that the standard symplectic form on $\mathbb{R}^{2n}$ is $\omega_0$. The key to proceeding from here (aside from knowing some identities related to the Lie derivative) is a result regarding the flow generated by a vector field. A vector field $X$ naturally defines an ordinary differential equation governing the behavior of some smooth curve $\gamma : (-a, a) \to \mathbb{R}^{2n}$ by forcing the vector tangent to $\gamma$ at $t$ to be $X_{\gamma(t)}$. The flow of the vector field $X$ is the one-parameter family of functions $F_t$ that each map “initial conditions” for the aforementioned differential equation, i.e. points on the manifold, to their solutions at time $t$. As it turns out, provided $t$ is small enough, $F_t$ is a diffeomorphism.

Keeping this in mind, define a family of 2-forms $\omega_t = t\omega_0 + (1 - t)\omega$. What we want is a diffeomorphism $F$ from an open subset of $\mathbb{R}^{2n}$ to an open subset of $U$ such that $F^*\omega = \omega_0$. If we assume there is some vector field $X$ with a flow that satisfies $F_t^*\omega = \omega_t$, then it turns out that’s enough to uniquely determine $X$ (see PlanetMath.org for details). Using properties of $X$, we can then choose a small enough neighborhood of a point on $U$ so that $F_t$ is a diffeomorphism for $t$ greater than $1$ (i.e. $t = 1$ meets the “small enough” criterion above). It’s then evident that $F_1$ provides us with the diffeomorphism we were looking for. Then the theorem is proved.