Math 4530 — First Midterm Exam — Brief solutions

1. (a) It suffices to show that if \( a \) and \( b \) are in \( X \) then there is a path from \( f(a) \) to \( f(b) \) in \( f(X) \).

   Well, as \( X \) is path connected, there is a path \( \gamma : [0,1] \to X \) from \( a \) to \( b \). As the composition of two continuous functions is again continuous, \( f \circ \gamma \) is a path in \( f(X) \), and it runs from \( f(a) \) to \( f(b) \).

   (b) Points on the sphere are parametrized by a pair of angles \( \theta \in \mathbb{R} \) and \( \phi \in (-\pi/2, \pi/2) \), via a continuous map from \( [0,2\pi] \times \mathbb{R} \) into \( S^2 \). As \( [0,2\pi] \times (-\pi/2, \pi/2) \) is path-connected, so is \( S^2 \).

2. This is “bookwork” — please look it up in lecture notes, or in a textbook!

3. (a) A subset \( A \) of a topological space \( X \) is connected when it is connected in the subspace topology on \( A \) — that is, whenever \( A \) is the disjoint union of two subsets \( U \) and \( V \) that are open in \( A \), one of \( U \) and \( V \) must be empty.

   (b) i. If \( A \) and \( B \) are connected subsets of a topological space and \( A \cap B \neq \emptyset \), then \( A \cup B \) is connected — this is a special case of a result proved in lectures.

   ii. If \( A \) is the line \( y = 0 \) in the plane and \( B \) is the circle \( S^1 \), then both \( A \) and \( B \) are connected, but their intersection is \( \{(1,0), (-1,0)\} \), which is not connected.

4. (a) The subset \( \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1 \text{ and } x > 0\} \) of \( \mathbb{R}^3 \) (with the usual topology) is not compact, by the Heine–Borel Theorem, since it is not closed.

   (b) The set \( \{1,2,\ldots,n\} \) with the discrete topology is compact since there are only finitely many open sets in the topology.

   (c) The subset \( \mathbb{R} \) of \( \mathbb{R} \) where \( \mathbb{R} \) has the topology that has basis \( \{(a,b) \mid a, b \in \mathbb{R}, a < b\} \) is not compact. For example \( [1,2] \) together with the sets \( [0,1 - 1/n] \) for \( n = 2, 3, \ldots \) form an open cover with no finite subcover.

   (d) The subset \( \mathbb{Z} \) of \( \mathbb{R} \) where \( \mathbb{R} \) has the topology in which sets are open when they are empty or have countable complement is not compact. For example, the sets \((\mathbb{R} \setminus \mathbb{Z}) \cup \{n\}\), with \( n \) ranging over \( \mathbb{Z} \), forms an open cover with no finite subcover.

TRR, October 2009