Math 4530 — First Midterm Exam
2:55pm-4:10pm, Thursday 1st October 2009

"The mathematician does not study pure mathematics because it is useful; he studies it because he delights in it and he delights in it because it is beautiful." Henri Poincaré.

Please answer all the questions and justify your answers. Use of calculators and other electronic devices is not permitted. Notes and books may not be used. Please write your name on every sheet you hand in. At the end of the exam you will be asked to number your pages.

1. (a) Suppose \( f : X \to Y \) is a continuous function between topological spaces. Prove that if \( X \) is path-connected, then \( f(X) \) is a path-connected subset of \( Y \).

(b) Use the result of part (a) to prove that the 2-sphere \( S^2 = \{ (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1 \} \) (with the subspace topology inherited from \( \mathbb{R}^3 \)) is path connected.

(3 + 3 pts)

2. (a) Define the subspace topology on a subset \( A \) of a topological space \( X \).

(b) Verify that the subspace topology satisfies the axioms for a topological space.

(3 + 3 pts)

3. (a) What does it mean to say a subset \( A \) of a topological space \( X \) is connected?

(b) Give a proof or counterexample for each of the following statements.

i. If \( A \) and \( B \) are connected subsets of a topological space and \( A \cap B \neq \emptyset \), then \( A \cup B \) is connected.

ii. If \( A \) and \( B \) are connected subsets of a topological space, then \( A \cap B \) is connected.

(2 + 3 + 3 pts)

4. Which of the following spaces are compact?

(a) The subset \{ \( (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1 \) and \( x > 0 \) \} of \( \mathbb{R}^3 \) (with the usual topology).

(b) The set \{1, 2, \ldots, n\} with the discrete topology.

(c) The subset \([0, 1]\) of \( \mathbb{R} \) where \( \mathbb{R} \) has the topology that has basis \{ \( [a, b] \mid a, b \in \mathbb{R}, \ a < b \) \}.

(d) The subset \( \mathbb{Z} \) of \( \mathbb{R} \) where \( \mathbb{R} \) has the topology in which sets are open when they are empty or have countable complement.

(2 + 2 + 2 + 2 pts)

(Total = 28 pts)

TRR, September 2009