PART 1

Answer ALL questions in this part.

1. TRUE or FALSE? Do NOT justify your answers.

(a) Every quotient of a Hausdorff space is Hausdorff.

(b) If $A$ and $B$ are deformation retracts of $C$, then $A$ and $B$ are homotopy equivalent.

(c) If $X$ is a connected topological space and $x_1, x_2 \in X$, then $\pi_1(X, x_1)$ is isomorphic to $\pi_1(X, x_2)$.

(d) If $E \to B$ is a covering map with $p(e) = b$ and $\pi_1(B, b)$ is abelian, then $\pi_1(E, e)$ is abelian.

(e) Every compact non-orientable surface without boundary is homeomorphic to a sphere with finitely many crosshandles and at most one crosscap.

(f) If $K$ is a knot in $\mathbb{R}^3$, then the abelianization of $\pi_1(\mathbb{R}^3 \setminus K)$ is $\mathbb{Z}$.

1 pt each = 6 pts
Incorrect answers will score 0.
2. What are the fundamental groups of the following spaces? Do NOT justify your answers.

(a) \( \mathbb{R}^3 \setminus \{(x,0,0) \mid x \in \mathbb{R}\} \).

(b) A Klein Bottle with a point removed.

(c) A cube with opposite faces identified — that is, the quotient \([0,1]^3/\sim\) where \((0,y,z) \sim (1,y,z), (x,0,z) \sim (x,1,z)\) and \((x,y,0) \sim (x,y,1)\) for all \(x,y,z \in [0,1]\), and no other points are identified.

(d) The space consisting of the 12 edges of a cube — that is, the subspace of \(\mathbb{R}^3\) consisting of all \((x,y,z)\) such that \(x,y,z \in [0,1]\) and at least two of \(x,y,z\) are in \(\{0,1\}\).

(2 pts each = 8 pts)

3. (a) Define the product topology and the box topologies on \(\prod_{i \in \mathbb{N}} \mathbb{R}\), the product of countably many copies of \(\mathbb{R}\) (with its usual topology).

(b) Show that the map \(f : \mathbb{R} \to \prod_{i \in \mathbb{N}} \mathbb{R}\), defined by \(f(z) = (z,z,z,\ldots)\), is continuous with respect to one of these topologies but not the other.

(2 + 2 + 2 + 2 = 8 pts)

4. What is the maximum number of disjoint non–self–intersecting loops that can be removed from a closed orientable surface of genus \(g\) without leaving more than one connected component? (Hint: use Euler characteristic.)

(8 pts)

(Total = 30 pts)
PART 2

Choose ONLY TWO questions from this part. If you attempt more than two questions, you must indicate which two you would like to be graded. Each question is worth 15 points.

1. (*One-point compactifications.*) Assume $X$ is a non–compact connected Hausdorff space in which every point has a compact neighbourhood. Define $X' := X \cup \{\infty\}$ — that is $X$ with one extra point called $\infty$ which is not in $X$. [*In this question you may use without proof the fact that the intersection of a family of compact sets is compact.*]

(a) For a subset $U \subset X$, say that $U$ is open in $X'$ when it is open in $X$, and that $U \cup \{\infty\}$ is open in $X'$ when $X \setminus U$ is compact in $X$. Show that this defines a topology on $X'$.
(b) Show that $X'$ is compact.
(c) Show that $X'$ is connected.
(d) Show that if $X$ is $\mathbb{R}^2$ with its usual topology, then $X'$ is homeomorphic to the 2–sphere.

(4 + 4 + 3 + 4 = 15 pts)

2. (a) Explain why (and how) a continuous map $f : X \to Y$ with $f(x) = y$ induces a homomorphism $\pi_1(X, x) \to \pi_1(Y, y)$

(b) Use the fact that $\pi_1(S^1) \cong \mathbb{Z}$ to prove Brouwer’s Fixed Point Theorem: for every continuous map $f : D^2 \to D^2$ there exists $a \in D^2$ such that $f(a) = a$.

(c) Show that if a space $X$ has the property that every continuous map $f : X \to X$ has a fixed point, then so does each retract $Y$ of $X$.

(5 + 7 + 3 = 15 pts)

3. (a) Suppose $X$ is a topological space and $X = U \cup V$ where $U$ and $V$ are open, $U \cap V$ is path–connected and non–empty, and $U$ and $V$ are both simply connected. Prove (without using the Seifert van Kampen Theorem) that $X$ is simply connected.

(b) Explain how part (a) is a special case of the Seifert van Kampen Theorem.

(c) Use the Seifert van Kampen Theorem to calculate the fundamental group of the surface obtained by gluing a disc to a Möbius band by identifying their boundary circles.

(7 + 2 + 6 = 15 pts)
4. (a) Let $E$ and $B$ be path–connected topological spaces. What does it mean to say that $p : E \to B$ is a covering map?

(b) Now suppose that $E$ is simply-connected, so that $\pi_1(E, e)$ is trivial where $e$ is a given point of $E$. Construct a bijection between the sets $\pi_1(B, b)$ and $p^{-1}(b)$ where $b = p(e)$.

[You may quote without proof theorems concerning the existence and uniqueness of liftings of maps $Z \to B$.]

(c) Write down (without proof) an example of a topological space $B$ and a simply–connected covering space $E$ with

i. $\pi_1(B, b)$ isomorphic to the direct product $C_2 \times C_2$ — in other words, the group presented by $\langle x, y \mid x^2, y^2, x^{-1}y^{-1}xy \rangle$.

ii. $\pi_1(B, b)$ isomorphic to the free product $\mathbb{Z} \ast \mathbb{Z}^2$ — in other words, the group presented by $\langle x, y, z \mid y^{-1}z^{-1}yz \rangle$.

(3 + 6 + 3 + 3 = 15 pts)

5. Suppose a group $G$ acts by homeomorphisms on a topological space $X$.

(a) What is meant by $X/G$? What is the “natural” (or “canonical”) map $p : X \to X/G$?

(b) Assume that the action is proper discontinuous — that is, for all $x \in X$, there exists an open subset $V \subseteq X$ with $x \in V$, such that $V^g \cap V^h = \emptyset$ for all distinct pairs of elements $g, h \in G$.

i. Why is $p$ continuous and surjective?

ii. Show that if $U \subseteq X$ is open then $p(U)$ is open.

iii. Show that $p$ is a covering map.

(c) Describe a group $G$ of symmetries of the plane $\mathbb{R}^2$ such that $\mathbb{R}^2/G$ is homeomorphic to the Klein bottle.

(3 + 1 + 3 + 3 + 5 = 15 pts)