Please answer all the questions and justify your answers. Use of calculators and other electronic devices is not permitted. Notes and books may not be used. Please write your name on your exam booklet.

1. Suppose $X$ and $Y$ are topological spaces.
   (a) Define the **product topology** on $X \times Y$. (You are not asked to prove it is a topology.)
   (b) Define the **subspace topology** on a subset $A$ of $Y$. (You are not asked to prove it is a topology.)
   (c) What does it mean to say a map $X \rightarrow Y$ is a **homeomorphism**?
   (d) Show that $X$ is homeomorphic to the subspace $\Delta := \{(x, x) \mid x \in X\}$ of $X \times X$.

\[4 + 4 + 4 + 4 = 16\text{ pts}\]

2. (a) What does it mean to say that a subset $V$ of a topological space $X$ is closed?
   (b) Show that if a set $C$ of subsets of a set $X$ satisfies
      - $\emptyset, X \in C$,
      - $A \cup B \in C$ for all $A, B \in C$, and
      - $\bigcap_{i \in I} C_i \in C$ for all subsets $\{C_i \mid i \in I\}$ of $C$ (here $I$ is any index set),
   then $C$ is the set of closed subsets for a topology on $X$.
   (c) *(The Zariski Topology)* Consider the following topology on $\mathbb{R}^2$: a set $C$ is closed if it is the zero set of a system of polynomials in two variables—that is,
      \[C = \{(x, y) \mid f_j(x, y) = 0 \text{ for all } j \in J\}\]
   for some family of polynomials $f_j(x, y)$ indexed by a set $J$. Verify that this indeed defines a topology on $\mathbb{R}^2$.

\[4 + 4 + 8 = 16\text{ pts}\]

3. (a) Suppose $\sim$ is an equivalence relation on a topological space $X$. What is meant by the **quotient topology** on $X/\sim$?
   (b) What does it mean to say a surjective map $f : X \rightarrow Y$ between topological spaces is a **quotient map**?
   (c) Suppose $f : X \rightarrow Y$ is a quotient map. Give, without proof, an equivalence relation on $X$ such that $X/\sim$ is homeomorphic to $Y$.
   (d) Consider $\mathbb{R}^2$ with its usual topology. Let $\sim$ be the equivalence relation on $\mathbb{R}^2$ such that $(a, b) \sim (c, d)$ if and only if $a^2 + b^2 = c^2 + d^2$. Show that $\mathbb{R}^2/\sim$ is homeomorphic to $[0, \infty)$.

\[4 + 4 + 4 + 4 = 16\text{ pts}\]

\[\text{(Total = 48 pts)}\]

TRR, October 2015