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What is amenability?

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"messbar" von Neumann, 1929

A group G is amenable if it admits a left–invariant, finitely additive probability measure μ.

$$\forall A \subseteq G, \quad \forall g \in G, \quad \mu(gA) = \mu(A).$$

Example

G finite,

$$\mu(A) = \frac{|A|}{|G|} .$$

Equivalently, there is a left–invariant mean $f \mapsto \int f \, d\mu$ on the space $L(G, \mathbb{R})$ of bounded functions $G \to \mathbb{R}$.

That is, $f \mapsto \int f \, d\mu$ is a map $L(G, \mathbb{R}) \to \mathbb{R}$ satisfying

- linearity
- $f \geq 0 \implies \int f \, d\mu \geq 0$
- $\int \chi_G \, d\mu = 1$
- $\forall f \in L(G, \mathbb{R}), \quad \forall g \in G, \quad \int (gf) \, d\mu = \int f \, d\mu$

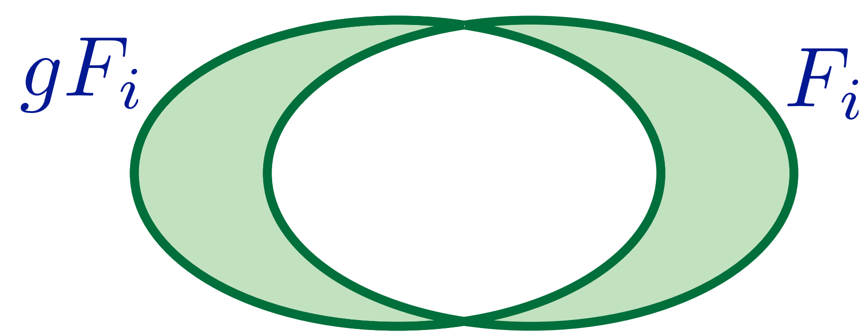
$(gf)(h) := f(g^{-1}h)$

$$\mu(A) = \int \chi_A \, d\mu$$

G a countable group

A **Følner sequence** for G is a sequence (F_i) of finite subsets such that

~~$\bigcup_i F_i = G$~~ and $\forall g \in G, \quad \lim_{i \to \infty} \frac{|gF_i \, \Delta \, F_i|}{|F_i|} = 0$.



Example. $F_i = \{-i, \ldots, i-1, i\}$ is a Følner sequence for $\mathbb{Z}$.

Proposition. G is amenable if and only if it admits a Følner sequence.

"if":

$$\mu(A) = \lim_{\mathcal{U}} \frac{|A \cap F_i|}{|F_i|}$$

$\mathcal{U}$ a non–principal ultrafilter on $\mathbb{N}$

A **paradoxical decomposition** of a group is a partition

$$G = U_1 \sqcup \cdots \sqcup U_r \sqcup V_1 \sqcup \cdots \sqcup V_s$$

with $g_1, \ldots, g_r, h_1, \ldots, h_s \in G$ such that

$$G = g_1 U_1 \sqcup \cdots \sqcup g_r U_r = h_1 V_1 \sqcup \cdots \sqcup h_s V_s .$$

Almost example. $G = F_2 = \langle x, y \rangle$

$$W(a) := \{ \text{reduced words beginning with } a \ \}$$

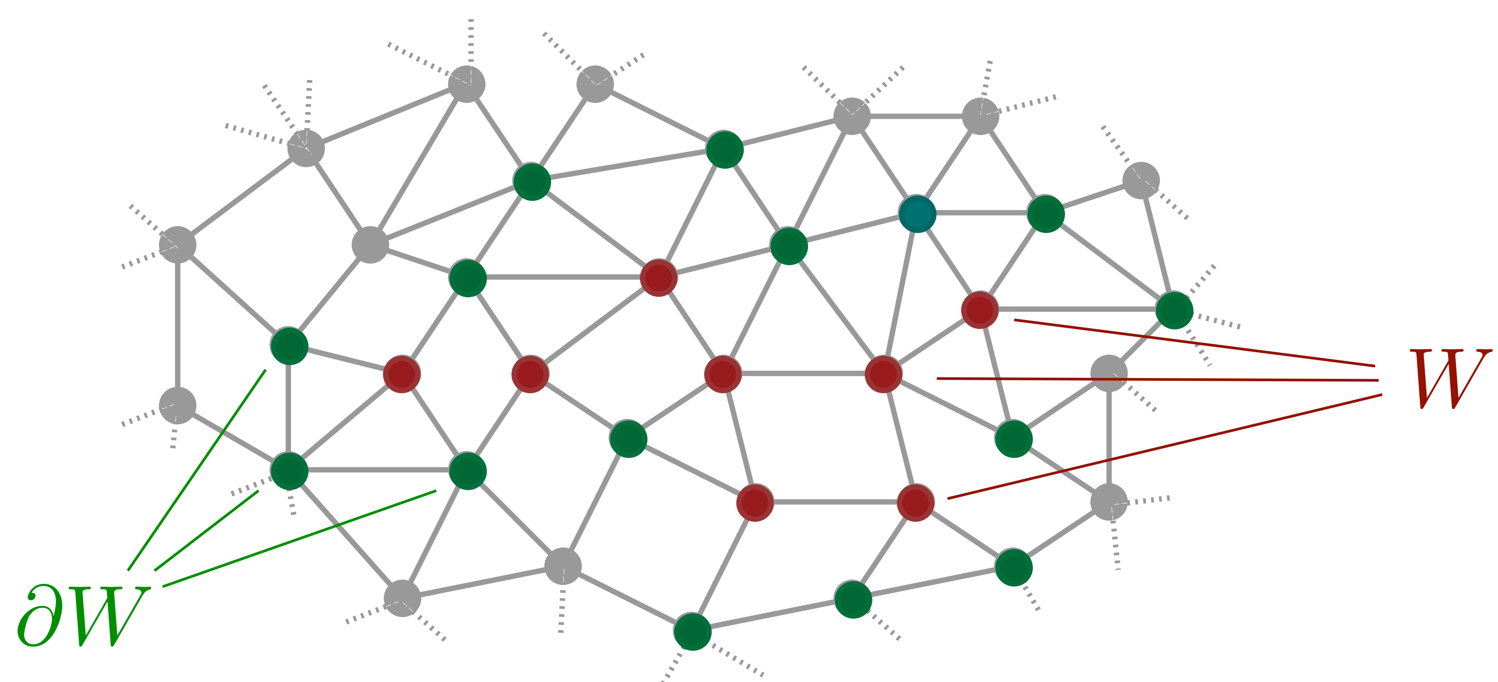
$$\begin{aligned} F_2 &= \{1\} \sqcup W(x) \sqcup W(x^{-1}) \sqcup W(y) \sqcup W(y^{-1}) \\ &= W(x) \sqcup xW(x^{-1}) = W(y) \sqcup yW(y^{-1}) \end{aligned}$$

→ Banach–Tarski Paradox

Proposition. G is amenable if and only if it has no paradoxical decomposition.

A graph enjoys a **strong isoperimetric inequality** when $\exists\ \varepsilon > 0$, such that if W is a finite set of vertices then

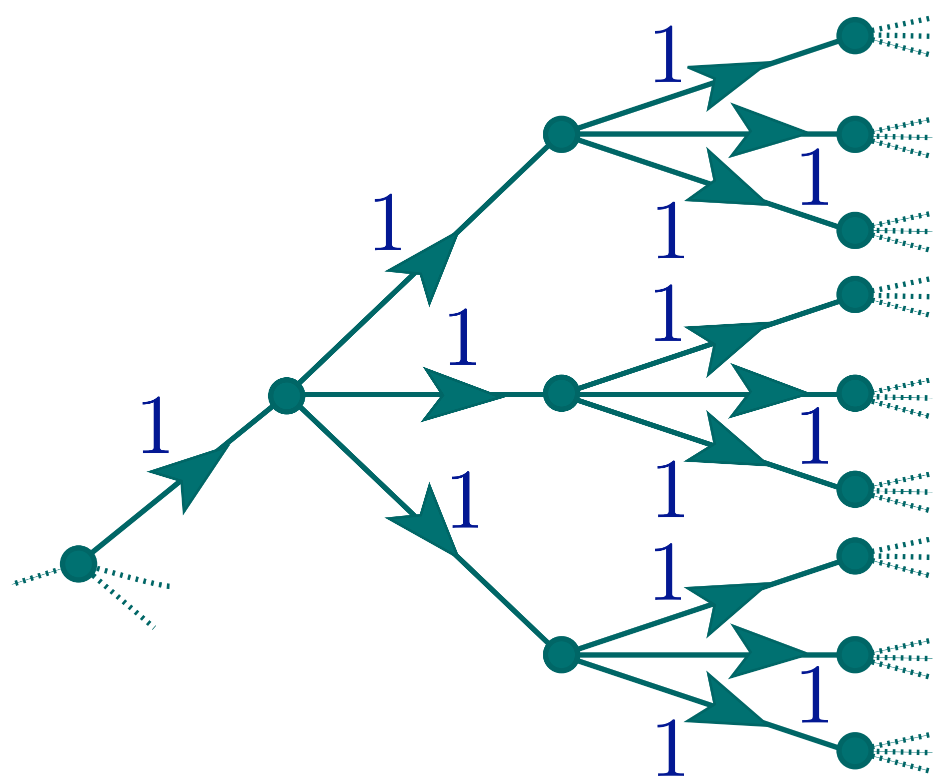
$$|\partial W| \geq \varepsilon |W| .$$



A flow on a graph is an assignment of directions and weights in $[0, \infty)$ to the edges.

The production at a vertex is the sum of the weights of the outgoing edges, minus the sum of the weights of the incoming edges.

Example.



Every vertex produces 2.

A flow is bounded if the weights are all $\leq K$ for some K.

A flow is uniformly productive if $\exists \varepsilon > 0$ such that every vertex produces $\geq \varepsilon$.

Example. 

No uniformly productive bounded flow.

G a group with finite generating set $\mathcal{A}$.

Characterisations of amenability

(1) G has a left–invariant finitely additive probability measure.

$\Updownarrow$

(2) $L(G, \mathbb{R})$ left–invariant mean.

$\Uparrow$

(3) G has a Følner sequence.

$\Uparrow$

(4) $\mathrm{Cay}(G, \mathcal{A})$ does not satisfy a strong isoperimetric inequality.

$\Uparrow$

(5) There is no uniformly productive bounded flow on $\mathrm{Cay}(G, \mathcal{A})$.

$\Uparrow$

(6) G has no paradoxical decomposition.

(7) The probability $\phi(n)$ that a uniform random walk on $\mathrm{Cay}(G, \mathcal{A})$ returns to its starting point after $2n$ steps, does not decay exponentially fast. (*Kesten's Criterion.*)

Amenability and growth

Finitely generated groups G of subexponential growth are amenable.

In fact, if $\mathcal{A}$ is a generating set and

$$B_n \ := \ (\mathcal{A}^{\pm 1} \cup \{1\})^n$$

then some subsequence of $(B_n)_{n\in\mathbb{N}}$ is a Følner sequence.

Proof. Subexponential growth implies

$$\forall \varepsilon > 0, \quad \exists k_\varepsilon, \quad |B_{k_\varepsilon+1}|/|B_{k_\varepsilon}| < 1+\varepsilon.$$

Let $n_i = k_{1/i}$. Then

$$\forall g \in \mathcal{A}, \quad \frac{|gB_{n_i}\Delta B_{n_i}|}{|B_{n_i}|} \ \leq \ \frac{2(|B_{n_i+1}| - |B_{n_i}|)}{|B_{n_i}|} \ < \ \frac{2}{i} \ \to \ 0.$$

It follows that

$$\forall g \in G, \quad \frac{|gB_{n_i}\Delta B_{n_i}|}{|B_{n_i}|} \ \to \ 0.$$

∎

Example. Finitely generated nilpotent groups are amenable.

Elementary amenable groups

- Subgroups and quotients of amenable groups are amenable.
- Amenable–by–amenable groups are amenable.
- Direct unions of amenable groups are amenable.

The class of elementary amenable groups is the smallest that contains all finite and abelian groups, and is closed under the above operations.

Examples

- Solvable groups are elementary amenable.
- Grigorchuk's group of intermediate growth is amenable but not elementary amenable.

If a group contains an F_2 subgroup, then it is not amenable.

von Neumann Conjecture. (*Day, 1957?*)
The converse holds.

False! (*Ol'shanskii, 1980*) There is an infinite non–amenable torsion group – a Tarski monster.

Outstanding open question.

Is Thompson's group F amenable?

- F has no rank–2 free subgroup.
- F is not elementary amenable.