Big numbers, graph coloring, and Hercules’ battle with the hydra

Tim Riley
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K–12 Education and Outreach
You have two minutes. Using standard math notation, English words, or both, name a single whole number—not an infinity—on a blank index card. Be precise enough for any reasonable modern mathematician to determine exactly what number you’ve named, by consulting only your card and, if necessary, the published literature.
The largest of all your numbers, plus one.
<table>
<thead>
<tr>
<th>Number of zeros</th>
<th>U.S. &amp; scientific community</th>
<th>Other countries</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>thousand</td>
<td>thousand</td>
</tr>
<tr>
<td>6</td>
<td>million</td>
<td>million</td>
</tr>
<tr>
<td>9</td>
<td>billion</td>
<td>1000 million (1 milliard)</td>
</tr>
<tr>
<td>12</td>
<td>trillion</td>
<td>billion</td>
</tr>
<tr>
<td>15</td>
<td>quadrillion</td>
<td>1000 billion</td>
</tr>
<tr>
<td>18</td>
<td>quintillion</td>
<td>trillion</td>
</tr>
<tr>
<td>21</td>
<td>sextillion</td>
<td>1000 trillion</td>
</tr>
<tr>
<td>24</td>
<td>septillion</td>
<td>quadrillion</td>
</tr>
<tr>
<td>27</td>
<td>octillion</td>
<td>1000 quadrillion</td>
</tr>
<tr>
<td>30</td>
<td>nonillion</td>
<td>quintillion</td>
</tr>
<tr>
<td>33</td>
<td>decillion</td>
<td>1000 quintillion</td>
</tr>
<tr>
<td>36</td>
<td>undecillion</td>
<td>sextillion</td>
</tr>
<tr>
<td>39</td>
<td>duodecillion</td>
<td>1000 sextillion</td>
</tr>
<tr>
<td>42</td>
<td>tredecillion</td>
<td>septillion</td>
</tr>
<tr>
<td>45</td>
<td>quattuordecillion</td>
<td>1000 septillion</td>
</tr>
<tr>
<td>48</td>
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<td>octillion</td>
</tr>
<tr>
<td>51</td>
<td>sexdecillion</td>
<td>1000 octillion</td>
</tr>
<tr>
<td>54</td>
<td>septendecillion</td>
<td>nonillion</td>
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<tr>
<td>57</td>
<td>octodecillion</td>
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</tr>
<tr>
<td>60</td>
<td>novemdecillion</td>
<td>decillion</td>
</tr>
<tr>
<td>63</td>
<td>vigintillion</td>
<td>1000 decillion</td>
</tr>
<tr>
<td>66 - 120</td>
<td></td>
<td>undecillion - vigintillion</td>
</tr>
<tr>
<td>303</td>
<td>centillion</td>
<td>centillion</td>
</tr>
<tr>
<td>600</td>
<td></td>
<td>centillion</td>
</tr>
</tbody>
</table>
A googol = \(10^{100}\)

A googolplex = \(10^{\text{googol}}\)
"γαμμηκοσιοί" (Greek)
= "sand-hundred"

Pindar: "sand escapes counting"

A vigintillion
$10^{63}$

"The Sand-Reckoner"

Pindar: ca. 522 – 443 BC

Archimedes
c. 287 BC – c. 212 BC
<table>
<thead>
<tr>
<th>Number</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$9.2 \times 10^{26}$</td>
<td>Approximate diameter (in metres) of the visible universe (92 billion light years).</td>
</tr>
<tr>
<td>$1 \times 10^{27}$</td>
<td>Temperature (in ° Kelvin) of the universe $10^{-35}$ seconds after the Big Bang, at the start of the inflationary epoch.</td>
</tr>
<tr>
<td>$2 \times 10^{30}$</td>
<td>Mass (in kilograms) of the Sun (1 solar mass).</td>
</tr>
<tr>
<td>$4 \times 10^{31}$</td>
<td>Mass (in kilograms) of Betelgeuse, a red supergiant star (about 20 solar masses).</td>
</tr>
<tr>
<td>$1.417 \times 10^{32}$</td>
<td>Planck Temperature, the temperature (in ° Kelvin) of the universe at 1 Planck Time after the Big Bang.</td>
</tr>
<tr>
<td>$1 \times 10^{40}$</td>
<td>Approximate ratio of the strength of the electromagnetic to the gravitational force between sub-atomic force particles.</td>
</tr>
<tr>
<td>$3.6 \times 10^{40}$</td>
<td>Mass (in kilograms) of OJ287, the largest measured supermassive black hole.</td>
</tr>
<tr>
<td>$6.87 \times 10^{41}$</td>
<td>Gravitational binding energy (in Joules) of the Sun.</td>
</tr>
<tr>
<td>$1.2 \times 10^{44}$</td>
<td>Estimated energy (in Joules) released in a supernova explosion.</td>
</tr>
<tr>
<td>$3 \times 10^{52}$</td>
<td>Estimated mass (in kilograms) of the observable universe.</td>
</tr>
<tr>
<td>$4 \times 10^{69}$</td>
<td>Estimated total mass-energy (in Joules) of the observable universe.</td>
</tr>
<tr>
<td>$1 \times 10^{80}$</td>
<td>Estimate the total number of fundamental particles in the observable universe (other estimates go up to $10^{85}$).</td>
</tr>
<tr>
<td>$5.1 \times 10^{96}$</td>
<td>Planck density, the density (in kg/metre$^3$) of the universe at one unit of Planck time after the Big Bang.</td>
</tr>
</tbody>
</table>
Exponential growth
Bozorgmehr, King Anushirvan of Persia's grand vizier, challenges the Indian envoy to a game of chess
Graph coloring and chromatic number

Petersen Graph

Groetzsch Graph

Clebsch Graph
Petersen Graph
chromatic number = 3

Clebsch Graph
chromatic number = 4

Groetzsch Graph
chromatic number = 4
Is there a polynomial time algorithm that will tell you whether a graph is 3-colorable?
- Subset sum
- Hamiltonian cycles
- Travelling salesman problem
Graham’s number and Ramsey Theory
Ramsey Theory game

“Sim”
Guatsavo Simmons 1969

$K_5$

$K_6$

$K_7$
On $K_5$, there can be draws.

$K_6$ – At a gathering of any six people, some three of them are either mutual acquaintances or complete strangers.

There are no draws.

On $K_6$, with perfect strategy, the second player always wins.
$R(m)$ is the minimal $n$ such that however one 2–colors $K_n$, it will always contain a monochrome $K_m$ subgraph.

\[
R(1) = 1 
R(2) = 2 
R(3) = 6 
R(4) = 18 \]
\[43 \leq R(5) \leq 49\]
\[102 \leq R(6) \leq 165\]

“Erdős asks us to imagine an alien force, vastly more powerful than us, landing on Earth and demanding the value of $R(5)$ or they will destroy our planet. In that case, he claims, we should marshal all our computers and all our mathematicians and attempt to find the value. But suppose, instead, that they ask for $R(6)$. In that case, he believes, we should attempt to destroy the aliens.”

Paul Erdős, 1913 – 1996
Consider an $n$-dimensional hypercube, and connect each pair of vertices to obtain $K_{2^n}$.

What is the smallest value of $n$ such that every 2-coloring contains at least one monochrome planar $K_4$ subgraph?
Knuth’s arrow notation

\[ a \uparrow b = a^b = \underbrace{a \times a \times \ldots \times a}_{b \text{ copies of } a} \]

\[ a \uparrow\uparrow\ldots\uparrow b = a \uparrow\ldots\uparrow a \uparrow\ldots\uparrow a \ldots a \uparrow\ldots\uparrow a \quad \text{evaluated right to left.} \]

Example

\[ 3 \uparrow\uparrow\uparrow 3 = 3 \uparrow\uparrow 3 \uparrow\uparrow 3 = 3 \uparrow\uparrow (3 \uparrow 3 \uparrow 3) = \underbrace{3 \uparrow 3 \uparrow 3 \text{ copies of } 3}_{3 \uparrow 3 \uparrow 3} \]

\[ = \underbrace{7,625,597,484,987 \text{ copies of } 3}_{3 \uparrow 3 \uparrow 3} \]
is an upper bound for the solution to the hypercube problem.
General belief is that the answer to Graham’s problem is 6.

“The infinite we shall do right away. The finite may take a little longer.”

Stanislaw Ulam
A *hydra* is a word on letters $a$ and $b$.

- Hercules strikes off the first letter.
- The hydra regenerates by:
  - $a \mapsto ab$
  - $b \mapsto b$

Repeat.

**Example**

```
baab
abbbabbb
bbabbabb
babbbabb
abaababb
babbabbab
bbaababb
babbbabb
babbabbab
bbaababb
babbbabb
```

- **Who wins?**
- **How long does it take?**
- **Generalisations?**  Questions?

Hercules is victorious in 12 strikes.
Hercules wins against all hydra.

$2^n - 1$ strikes
Hydra using three letters \(a, b\) and \(c\).

Regeneration: \(a \mapsto ab, \; b \mapsto bc, \; c \mapsto c\).

<table>
<thead>
<tr>
<th>(a)</th>
<th>(aa)</th>
<th>(aa a)</th>
<th>(aa a a)</th>
<th>(aa a a a)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>(ab)</td>
<td>(abc)</td>
<td>(abcc)</td>
<td>(abc)</td>
</tr>
<tr>
<td>(b)</td>
<td>(bc)</td>
<td>(bccc)</td>
<td>(bccc)</td>
<td>(bccc)</td>
</tr>
<tr>
<td>(c)</td>
<td>(c)</td>
<td>(c)</td>
<td>(c)</td>
<td>(c)</td>
</tr>
</tbody>
</table>

\[\mathcal{H}(w) := \text{# strikes it takes to kill } w.\]

\[\mathcal{H}(a^{n+1}) = 3 \cdot 2^{\mathcal{H}(a^n)} - 2\]

\(\mathcal{H}(w)\) for \(w = a^{46}\) gives 211106232532990 strikes, eventually killing all hydra!

46 strikes
Hydra using four letters $a$, $b$, $c$ and $d$.

Regeneration: $a \mapsto a\,b$, $b \mapsto b\,c$, $c \mapsto c\,d$, $d \mapsto d$.

<table>
<thead>
<tr>
<th>1 strike</th>
<th>5 strikes</th>
<th>3 \cdot 2^3 \cdot 2^95 - 1 - 1 - 1 strikes</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>$a,a,a$</td>
<td>Hercules still always wins!</td>
</tr>
<tr>
<td>$a,b$</td>
<td>$a,b,a,b$</td>
<td></td>
</tr>
<tr>
<td>$b,c$</td>
<td>$b,c,a,b,b,c$</td>
<td></td>
</tr>
<tr>
<td>$c,d$</td>
<td>$b,c,c,d,a,b,b,c,c,d$</td>
<td></td>
</tr>
<tr>
<td>$d$</td>
<td>$c,d,c,d,d,a,b,b,c,b,c,c,d,c,d,d$</td>
<td></td>
</tr>
</tbody>
</table>
\[ a_1, a_2, \ldots \]
\[ a_i \mapsto a_i a_{i-1}, \forall i > 1 \]
\[ a_1 \mapsto a_1 \]

\[ \mathcal{H}_k(n) := \mathcal{H}(a_k^n) < \infty \]

<table>
<thead>
<tr>
<th>(k)</th>
<th>(n)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>(\ldots)</th>
<th>(n)</th>
<th>(\ldots)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>(\ldots)</td>
<td>(n)</td>
<td>(\ldots)</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>1</td>
<td>3</td>
<td>7</td>
<td>15</td>
<td>(\ldots)</td>
<td>(2^n - 1)</td>
<td>(\ldots)</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>1</td>
<td>4</td>
<td>46</td>
<td>(211106232532990)</td>
<td>(\ldots)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>1</td>
<td>5</td>
<td>(3 \cdot 2^{3 \cdot 2^3 \cdot 29^5 - 1} - 1 - 1)</td>
<td>(\vdots)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>(\vdots)</td>
<td>(\vdots)</td>
<td>(\vdots)</td>
<td>(\vdots)</td>
<td>(\vdots)</td>
<td>(\vdots)</td>
<td>(\vdots)</td>
</tr>
</tbody>
</table>
**Ackermann’s function.** For integers $k, n > 0$,

$$A_1(n) := 2n \quad \text{and} \quad A_{k+1}(n) := A_k^n(1).$$

<table>
<thead>
<tr>
<th>$k$</th>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>...</th>
<th>$n$</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>8</td>
<td>10</td>
<td>...</td>
<td>$2n$</td>
<td>...</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>16</td>
<td>32</td>
<td>...</td>
<td>$2^n$</td>
<td>...</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>4</td>
<td>16</td>
<td>65536</td>
<td>$2^{65536}$</td>
<td>...</td>
<td>$2^{2^{n-1}}$</td>
<td>...</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>4</td>
<td>65536</td>
<td>$A_3(65536)$</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td></td>
</tr>
</tbody>
</table>

\[\forall k \geq 3, n \geq 2, \quad \mathcal{H}_k(n) \geq A_k(n)\]
\[\forall k \geq 1, n \geq 0, \quad \mathcal{H}_k(n) \leq A_k(n + k)\]
Theorem

\[ \langle a_1, \ldots, a_k, t, p \mid t^{-1} a_1 t = a_1, \]
\[ t^{-1} a_i t = a_i a_{i-1} \quad (i > 1) \]
\[ [p, a_i t] = 1 \quad (i > 0) \]\n
\[ = \langle G_k, p \mid [p, H_k] \rangle \] has Dehn function \( \simeq A_k \) when \( k > 1 \).
Going faster!

\[ n \mapsto A_n(n) \] is “recursive” but not “primitive recursive”.

\[ A_{\ldots\ldots}(\ldots\ldots) \]

where \( A_k(n) \) is Ackermann’s function.

Wilhelm Ackermann, 1896 – 1962
Bump the base and subtract 1

\[ 266 = 2^8 + 2^3 + 2 \]
\[ 6590 = 3^8 + 3^3 + 2 \]
\[ 65601 = 4^8 + 4^3 + 1 \]
\[ 390750 = 5^8 + 5^3 \]
\[ 1679831 = 6^8 + 5 \times 6^2 + 5 \times 6 + 5 \]
\[ 5765085 = 7^8 + 5 \times 7^2 + 5 \times 7 + 4 \]
\[ \vdots \]
\[ ? \]
\[ ? \]
Goodstein sequences

\[ 266 = 2^8 + 2^3 + 2 = 2^{2^2+1} + 2^{2+1} + 2 \]

\[ 443426488243037769948249630619149892886 = 3^{3^{3+1}} + 3^{3+1} + 2 \]

\[ 32317006071311007300714876686699519604441026697154840321303454275246551388678 \]

\[ 908931972014115229134636887179609218980194941195591504909210950881523864482831 \]

\[ 20630877367309960917501977503896521067960576383840675682767922186426197561618 \]

\[ 38094384761704705816458520363050428875758915410658086075523991239303855219143 \]

\[ 333896683424206849747865645694948561760353263220580778056593310261927084603141 \]

\[ 50258592864177116725943603718461857357598351152301659044036976132332872312271 \]

\[ 256847108202097251571017269313234696785425806566979350459972683529986382155251 \]

\[ 66389437335543602135433229604645318478604952148193555853611059596231681 \]

\[ 5^{5^{5+1}} + 5^{5+1} \]

\[ 6^{6^{6+1}} + 5 \cdot 6^6 + 5 \cdot 6^5 + \cdots + 5 \cdot 6 + 5 \]

\[ \cdots \]

\[ \cdots \]

Reuben Louis Goodstein
1912 – 1985
Raymond Smullyan
Balls in a box.

A game of “bounded height, but unbounded width”.
Gödel's First Incompleteness Theorem

Any effectively generated theory capable of expressing elementary arithmetic cannot be both consistent and complete. In particular, for any consistent, effectively generated formal theory that proves certain basic arithmetic truths, there is an arithmetical statement that is true, but not provable in the theory (*Kleene* 1967, p. 250).

Kirby & Paris (1982)
It is unprovable in Peano Arithmetic that all Goodstein sequences terminate at zero.
Turing Machines and Busy Beaver Functions

Alan Turing, 1912 – 1954
Radó’s Busy Beaver function:

\[ BB(n) \] is the maximum halting time of all halting Turing machines of size at most \( n \).
$BB(11111)$

where $BB(n)$ is the busy beaver function.
The largest whole number
nameable with 1,000
characters of English text.

One plus the largest whole number
nameable with 1,000 characters
of English text.

“The Berry Paradox”
G. G. Berry, 1867 – 1928
References / Acknowledgements / Further reading

- Scott Aaronson, *Who can name the bigger number?*  
  www.scottaaronson.com/writings/bignumbers.html

- Archimedes, *The Sand Reckoner*

- Will Dison and Timothy Riley, *Hydra groups*, front.math.ucdavis.edu/1002.1945


- Wikipedia – especially the articles on *Ramsey Theory, Ramsey’s Theorem, Graham’s number* and *Knuth’s arrow notation.*

Slides available at: www.math.cornell.edu/~riley/talks.html