Enumeration of Symmetry Classes of Parallelogram Polyominoes*

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Abstract

Parallelogram polyominoes form a subclass of convex polyominoes in the square lattice; they have been studied extensively in the literature. The enumeration of (translation-type) parallelogram polyominoes with respect to their perimeter and area was carried out first by Pólya, with further refinements by Bender, Klarner, Rivest, Delest, Fédou, Viennot, and Bousquet-Mélou. In this paper, we enumerate congruence classes of parallelogram polyominoes by counting orbits under the action of a subgroup $\mathfrak{D}_2$ of the dihedral group $\mathfrak{D}_4$, of symmetries of the square, generated by a 180° rotation and a diagonal reflection. This involves the enumeration of the symmetry classes of parallelogram polyominoes under cyclic subgroups of $\mathfrak{D}_2$. Asymmetric parallelogram polyominoes are also enumerated using Möbius inversion in the lattice of subgroups of $\mathfrak{D}_2$. This work extends similar results obtained in [18] for the class of convex polyominoes.

Résumé

Les polyominos parallélogrammes forment une sous-classe des polyominos convexes sur le réseau carré; cette classe a été étudiée en détail dans la littérature. Le dénombrement de ces polyominoes, à translations près, selon le périmètre et l'aire, a été réalisée d'abord par Pólya, puis a raffinée par Bender, Klarner, Rivest, Delest, Fédou, Viennot et Bousquet-Mélou. Dans cet article, nous dénombrons les classes de congruences de polyominos parallélogrammes, c'est-à-dire les orbites de polyominos parallélogrammes sous l'action d'un sous-groupe $\mathfrak{D}_2$ du groupe diédral $\mathfrak{D}_4$ (le groupe des symétries du carré), engendré par une rotation de 180° et une réflexion diagonale. Ceci fait intervenir l'énumération des classes de symétries de polyominos parallélogrammes correspondant aux sous-groupes cycliques de $\mathfrak{D}_2$. Nous dénombrons également les polyominos parallélogrammes asymétriques à l'aide de l'inversion de Möbius dans le treillis des sous-groupes de $\mathfrak{D}_2$. Ce travail étend des résultats similaires obtenus dans [18] pour la classe des polyominos convexes.

AMS subject classification: 05B50 (Primary), 05A15, 05A16, 05A30 (Secondary)

1 Introduction

Parallelogram polyominoes, sometimes called staircase polyominoes, form a subclass of (horizontally and vertically) convex polyominoes on the square lattice, characterized by the fact that they touch the bottom-left and the top-right corners of their minimal bounding rectangle. See Figure 1. A 90° rotation of these would give a distinct but equivalent class of parallelogram polyominoes. In the same way as for general convex polyominoes, the area of a parallelogram polyomino is defined as the number of cells that it contains and the half-perimeter is equal to the sum of its width and height. Considerable literature can be found on the enumeration of various classes of polyominoes having some convexity and directedness property, with motivation coming from combinatorics, statistical physics, computer science and recreational mathematics. See M. Bousquet-Mélou [5] for a recent survey. In particular, parallelogram polyominoes have been studied with respect to their perimeter and area first by Pólya, with further contributions yielding refined enumerations from Bender, Klarner, Rivest, Delest, Fédou, Viennot and others. Mireille Bousquet-Mélou, using the Temperley methodology ([25]), has given a generating function with respect to height, width, area and height of first and last columns ([3], [5]).

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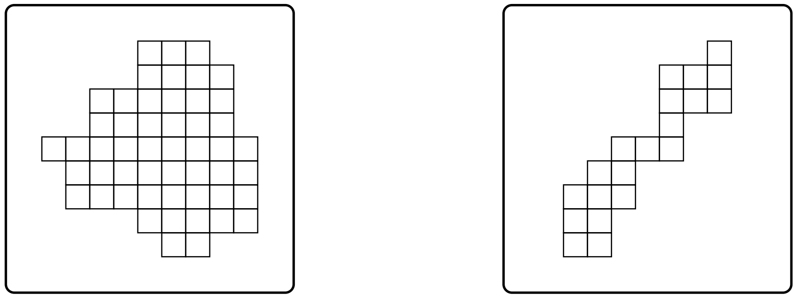

Figure 1: Convex (left) and parallelogram (right) polyominoes.

Polyominoes are usually considered equivalent if they can be obtained from one another by a plane translation. They are sometimes called translation-type polyominoes to be more precise (see D. A. Klarner [15]). It is natural to consider also *congruence-type* polyominoes, that is, equivalence classes of polyominoes under rotations and reflections. They occur as pieces that can freely move in space, as in plane packing problems (see S. W. Golomb [14]). In [18], the enumeration of congruence-type polyominoes, according to area and perimeter, has been carried out in the case of convex polyominoes. Redelmeier [21] enumerated general polyominoes according to their symmetries up to area 24 using a computer algorithm.

The problem is equivalent to the enumeration of orbits of the dihedral group $\mathfrak{D}_4$, of symmetries of the square, acting on convex polyominoes. The group $\mathfrak{D}_4$ contains eight elements, usually represented as 1, r, r^2, r^3, h, v, d_1 and d_2, where 1 denotes the identity element, r denotes a rotation by a right angle, h and v, reflections with respect to the horizontal and vertical axes respectively, and d_1 and d_2, reflections about the two diagonal axes of the square (we take the bissector of the first quadrant for d_2). The number of orbits $|X/G|$ of any finite group G acting on a set X is given by the Cauchy-Frobenius formula (alias Burnside's Lemma):

$$|X/G| = \frac{1}{|G|}\sum_{g\in G} |\mathrm{Fix}(g)|\,, \tag{1}$$

where $\mathrm{Fix}(g)$ denotes the set of elements of X which are g-symmetric, that is, invariant under g. Hence the enumeration of congruence-type convex polyominoes involves determining the size of the symmetry classes of convex polyominoes for each group element $g \in \mathfrak{D}_4$. Formula (1) is valid for infinite sets provided a weighted cardinality $|X|_\omega$ is taken, with respect to some G-invariant weight function ω. For a class $\mathcal{P}$ of polyominoes this means using generating series $\mathcal{P}(t, q)$ with respect to half-perimeter and area (variables t and q), for example.

The main goal of this paper is to carry out a similar procedure for the class $\mathbb{P}$ of parallelogram polyominoes. We observe that a subgroup of $\mathfrak{D}_4$ acts on parallelogram polyominoes, which we denote $\mathfrak{D}_2$, namely $\mathfrak{D}_2 = \langle r^2, d_1\rangle = \{1, r^2, d_1, d_2\}$, and that congruence types of parallelogram polyominoes coincide with orbits of $\mathbb{P}$ under $\mathfrak{D}_2$. In the following sections we therefore compute the generating series of the symmetry classes $\mathrm{Fix}(g)$ of parallelogram polyominoes for all $g \in \mathfrak{D}_2$ except the identity. We then use (1) to obtain $(\mathbb{P}/\mathfrak{D}_2)(t, q)$.

It is also possible to count asymmetric parallelogram polyominoes, that is polyominoes that are not g-invariant for any g except the identity, using Möbius inversion in the lattice of subgroups of $\mathfrak{D}_2$. This requires also the enumeration of the subclass $\mathrm{Fix}(\mathfrak{D}_2)$ of $\mathbb{P}$, of totally symmetric parallelogram polyominoes. We carry out this computation and show that asymmetric parallelogram polyominoes are asymptotically equivalent to all parallelogram polyominoes, as expected.

As we will see, the enumeration of all the symmetry classes of parallelogram polyominoes, according to perimeter, involves in one way or the other either the Dyck paths (or Dyck words, see J. Labelle [17]), counted by the Catalan numbers c_n, or the left factors of Dyck paths, counted by the central binomial coefficients b_n (see Cori and Viennot [8]), where

$$b_n = \binom{2n}{n} \qquad \text{and} \qquad c_n = \frac{1}{n+1}\binom{2n}{n}\,. \tag{2}$$

When the area is taken into account, q-analogues (some well-known and some novel) of these numbers appear naturally.

We would like to thank X. G. Viennot and M. Bousquet-Mélou for useful discussions.

2 Enumeration of parallelogram polyominoes

It has been known for a long time (Levine [19], Pólya [20]) that the number of parallelogram polyominoes of perimeter $2n$ is given by the Catalan number $c_{n-1} = \frac{1}{n}\binom{2n-2}{n-1}$. One proof of this fact is provided by the following bijection, due to Delest and Viennot ([11]) between parallelogram polyominoes of perimeter $2n+2$ and Dyck paths of length $2n$: given a parallelogram polyomino P of perimeter $2n+2$, let $(a_1, a_2, \ldots, a_k)$ be the sequence of column heights of P, and $(b_1, b_2, \ldots, b_{k-1})$ be such that b_i is the number of cells of contact between columns i and $i+1$ of P. The associated Dyck path D is the unique Dyck path with k peaks and $k-1$ valleys such that the peak heights are given in order by the sequence $(a_1, a_2, \ldots, a_k)$, and the valley heights by the sequence $(b_1 - 1, b_2 - 1, \ldots, b_{k-1} - 1)$ (the horizontal axis is at level 0). The height of P is $n + 1 - k = n - (k-1)$, which is also given by

$$a_1 + (a_2 - b_1) + (a_3 - b_2) + \ldots + (a_k - b_{k-1}) = \sum_{i=1}^{k} a_i - \sum_{j=1}^{k-1} b_j \,.$$

On the other hand, the number of $\searrow$ steps in D, that is the half-length of the path, is given by

$$\sum_{i=1}^{k} a_i - \sum_{j=1}^{k-1} (b_j - 1) = \sum_{i=1}^{k} a_i - \sum_{j=1}^{k-1} b_j + (k-1)\,,$$

which is seen to be n by the previous equation. Hence D is a Dyck path of length $2n$.

Also, note that the sum of the heights of the peaks, $\sum_{i=1}^{k} a_i$ is simply the area of P. Figure 2 illustrates the bijection.

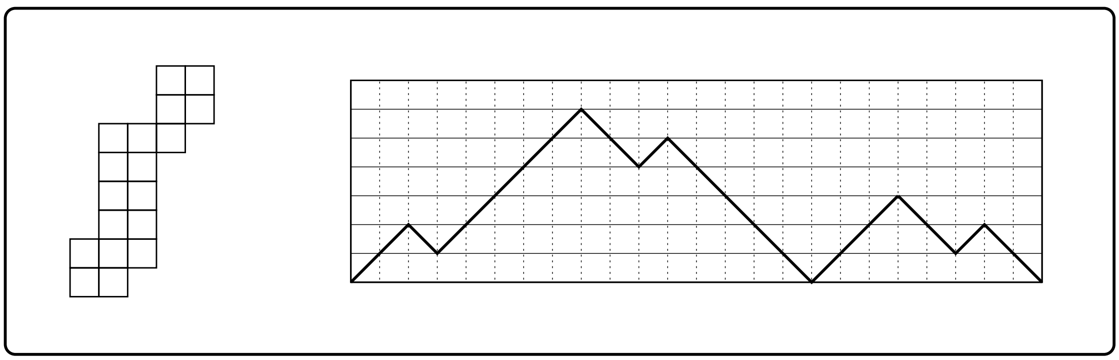

Figure 2: Parallelogram polyomino and associated Dyck path.

It follows that the generating series $\mathbb{P}(t)$ for parallelogram polyominoes according to half-perimeter is

$$\mathbb{P}(t) = \sum_{n \geq 2} c_{n-1} t^n = \frac{1 - 2t - \sqrt{1-4t}}{2}\,. \tag{3}$$

It has also been known for some time that when the area is taken into account, the generating series involves a quotient of q-analogues of Bessel functions (see Klarner and Rivest [16] and Bender [1]). Pólya ([20] and [12]) found a Laurent series relating the area and perimeter generating function to a specialization of itself, from which the terms of the series can be extracted easily. The width, height and area generating series can also be expressed as a continued fraction (see [13]). We will use the following recent more general form due to M. Bousquet-Mélou [3], giving the generating series $\mathbb{P}(v,x,y,q)$ of parallelogram polyominoes, where the variables v, x, y and q mark respectively the height of the rightmost column, the width, the overall height, and the area.

Proposition 1 ([3]) *The generating function $\mathbb{P}(v,x,y,q)$ of parallelogram polyominoes is given by*

$$\mathbb{P}(v,x,y,q) = vy\frac{J_1(v,x,y,q)}{J_0(x,y,q)}\,, \tag{4}$$

with

$$J_0(x,y,q) = \sum_{n\geq 0} \frac{(-1)^n x^n q^{\binom{n+1}{2}}}{(q)_n (yq)_n} \tag{5}$$

and

$$J_1(v,x,y,q) = \sum_{n\geq 1} \frac{(-1)^{n-1} x^n q^{\binom{n+1}{2}}}{(q)_{n-1}(yq)_{n-1}(1-vyq^n)} \tag{6}$$

with the usual notation $(a)_n = (a;q)_n = \prod_{i=0}^{n-1}(1-aq^i)$. ■

Note that the half-perimeter and area generating function $\mathbb{P}(t,q)$ of parallelogram polyominoes is obtained by putting $v=1$, $x=t$, $y=t$ in (4).

3 Symmetry classes of parallelogram polyominoes

3.1 Rotational symmetry

Observe that if we apply the Delest-Viennot bijection to an r^2-symmetric parallelogram polyomino of perimeter $2k+2$, the Dyck path we obtain is vertically symmetric (or, equivalently, the Dyck word associated to it is a palindrome). Hence we need only consider half the path, which is simply a left factor, of length k, of a Dyck path.

Proposition 2 *The number of r^2-symmetric parallelogram polyominoes of half-perimeter $k+1$ is equal to the number of left factors of Dyck paths, of length k.* ■

Corollary 3 *The number of r^2-symmetric parallelogram polyominoes of half-perimeter $k+1$ is given by*

$$r_{k+1}(1) = \begin{cases} \binom{k}{k/2} & \text{if } k \text{ is even}, \\ \frac{1}{2}\binom{k+1}{(k+1)/2} & \text{if } k \text{ is odd}. \end{cases} \tag{7}$$

Proof It is known that the number of left factors of length $2n$ of Dyck words is equal to the number of words in the alphabet {0,1} with distribution $0^n 1^n$, from which (7) follows easily. See [8] for a bijective proof. Here we prove (7) using generating functions. Dyck paths and left factors of Dyck paths are generated by the algebraic grammar

$$\begin{aligned} C &\to \varepsilon + xC\bar{x}C \\ L &\to C + CxL\,, \end{aligned}$$

over the alphabet $\{x,\bar{x}\}$. C denotes the Dyck paths and L the left factors, while x and $\bar{x}$ respectively denote a ↗ step and a ↘ step. The first production rule gives $C(x,\bar{x}) = 1 + x\bar{x}C(x,\bar{x})^2$, which we solve for

$$C(x,\bar{x}) = \frac{1-\sqrt{1-4x\bar{x}}}{2x\bar{x}}\,.$$

The second production rule gives $L(x,\bar{x}) = C(x,\bar{x})(1 + xL(x,\bar{x}))$, which we can solve, now that we have $C(x,\bar{x})$, for

$$L(x,\bar{x}) = \frac{\sqrt{1-4x\bar{x}}-1}{x(1-2\bar{x}-\sqrt{1-4x\bar{x}})}\,.$$

Substituting $x \mapsto t, \bar{x} \mapsto t$ into $L(x,\bar{x})$ gives the generating series $L(t)$ of left factors of Dyck paths by their length:

$$L(t) = \frac{2t-1+\sqrt{1-4t^2}}{2t(1-t)}\,,$$

from which (7) follows. ■

In order to include the area, we could extend another bijection, due to Bousquet-Mélou and Viennot [7], involving heaps of segments, to left factors of Dyck paths. However, there is a more direct approach. Indeed, parallelogram polyominoes with rotational symmetry can be obtained from two copies of a same parallelogram polyomino glued together. The glueing process depends on whether we want the final object to be of even width or of odd width, as can be seen in Figure 3. If $R_2(x,y,q)$ is the generating function of r^2-symmetric parallelogram polyominoes, then

$$R_2(x,y,q) = R_2^{(e)}(x,y,q) + R_2^{(o)}(x,y,q) \tag{8}$$

where $R_2^{(e)}(x,y,q)$ and $R_2^{(o)}(x,y,q)$ are respectively the generating series of even-width and odd-width r^2-symmetric parallelogram polyominoes.

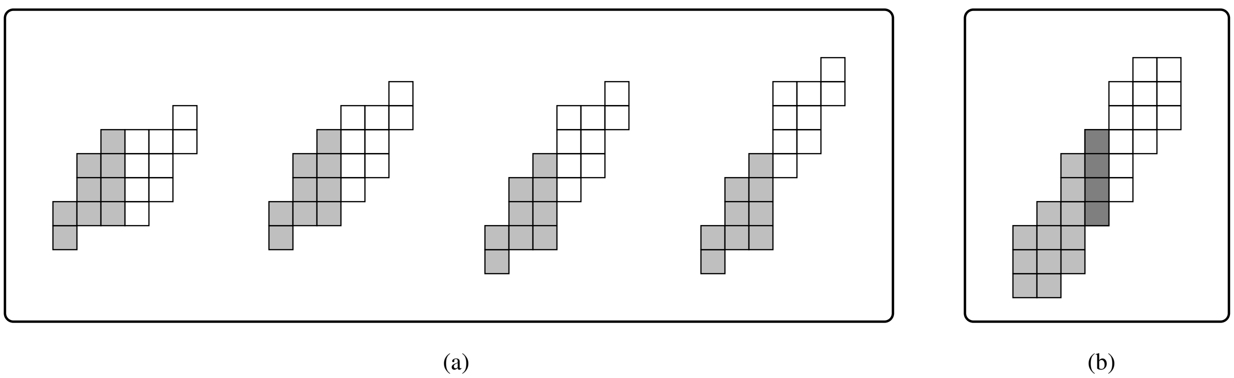

Figure 3: r^2-symmetric parallelogram polyominoes with (a) even width and (b) odd width.

Proposition 4 *The generating function $R_2^{(e)}(x,y,q)$ of even-width parallelogram polyominoes is given by*

$$R_2^{(e)}(x,y,q) = \frac{1}{1-y}\left(\mathbb{P}(\frac{1}{y},x^2,y^2,q^2) - \mathbb{P}(1,x^2,y^2,q^2)\right)\,. \tag{9}$$

where $\mathbb{P}(v,x,y,q)$ is the generating function (4) *of parallelogram polyominoes.*

Proof Let P be an r^2-symmetric parallelogram of even width. We define the fundamental region of P to be the left half of P. Call this polyomino Q (see Figure 3(a)). We first remark that Q is a parallelogram polyomino. To get P from Q, we rotate a copy of Q by 180° and glue the result $\overline{Q}$ to Q along the rightmost column. If this column has length equal to k, there will be k possible positions for $\overline{Q}$ relative to Q. The substitution $v \mapsto 1/y$, $x \mapsto x^2$, $y \mapsto y^2$ and $q \mapsto q^2$ in the generating series $\mathbb{P}(v,x,y,q)$ of parallelogram polyominoes corresponds to the highest position of $\overline{Q}$, which minimizes the overall height of P. All the possible positions will be accounted for by multiplying by $(1 + y + \ldots + y^{k-1})$. In other words, the substitution to make in $\mathbb{P}(v,x^2,y^2,q^2)$ is

$$v^k \mapsto \frac{1+y+\ldots+y^{k-1}}{y^k} = \frac{1}{1-y}\left(\frac{1}{y^k}-1\right)\,. \tag{10}$$

Summing over all possible k's, we find the proposed generating series (9) for r^2-symmetric parallelograms. ∎

Proposition 5 *The generating function $R_2^{(o)}(x,y,q)$ of odd-width parallelogram polyominoes is given by*

$$R_2^{(o)}(x,y,q) = \frac{1}{x}\mathbb{P}(\frac{1}{yq}, x^2, y^2, q^2)\,. \tag{11}$$

where $\mathbb{P}(v,x,y,q)$ is the generating function of parallelogram polyominoes.

Proof The proof is similar to the previous one. The main difference is that only one glueing position of $\overline{Q}$ to Q is admissible and that furthermore the rightmost column of Q and its rotated image in $\overline{Q}$ are superimposed to yield an odd width (see Figure 3(b)). Details are left to the reader. ∎

We would like to find the number of r^2-symmetric parallelograms of a given half-perimeter, without losing the area information, i.e. we want to express the generating series in the form

$$R_2(t,q) = R_2(t,t,q) = \sum_{k\geq 2} r_k(q)t^k\,. \tag{12}$$

The above expressions for the generating series of r^2-symmetric parallelogram polyominoes can be used to extract the polynomials $r_k(q)$ from it (i.e. develop it in powers of t after substituting $x \mapsto t, y \mapsto t$ in it). Here are the first few of these polynomials:

$$\begin{aligned}
r_2(q) &= q\\
r_3(q) &= 2q^2\\
r_4(q) &= q^4 + 2q^3\\
r_5(q) &= 2q^6 + 4q^4\\
r_6(q) &= q^9 + 2q^8 + q^7 + 2q^6 + 4q^5
\end{aligned}$$

3.2 Reflective symmetries

We begin by introducing a subfamily of parallelogram polyominoes which we will call *Dyck polyominoes* as they correspond to Dyck paths drawn over and above the main diagonal. We will also consider truncated Dyck polyominoes, which we will call *left factors of Dyck polyominoes* (or LFD polyominoes for short), again in analogy with the left factors of Dyck paths. Dyck and LFD polyominoes are illustrated in Figure 4.

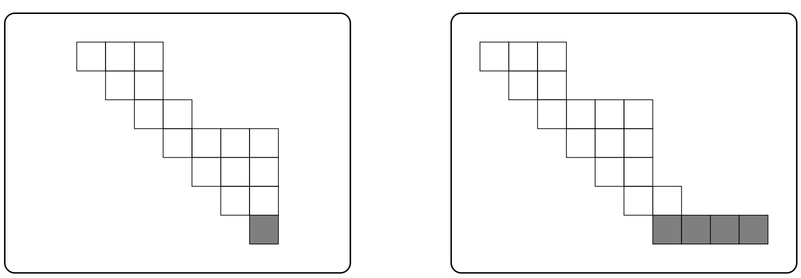

Figure 4: Dyck polyomino (left) and left factor of Dyck polyomino (right).

We introduce $L_n(u) = L_n(u,y,q)$ the generating function for LFD polyominoes having a basis of width n, with variables u, y and q corresponding to the number of cells of the uppermost row, the height and the area respectively. $L_n(u)$ can be defined recursively by the following functional equation, illustrated in Figure 5:

$$L_n(u) = u^n yq^n + \frac{yu^2q^2}{1-uq}(L_n(1) - L_n(uq))\,. \tag{13}$$

The generating function $L(u)$ of all LFD polyominoes is simply the sum over all possible base widths,

$$L(u) = \sum_{n\geq 1} L_n(u)\,. \tag{14}$$

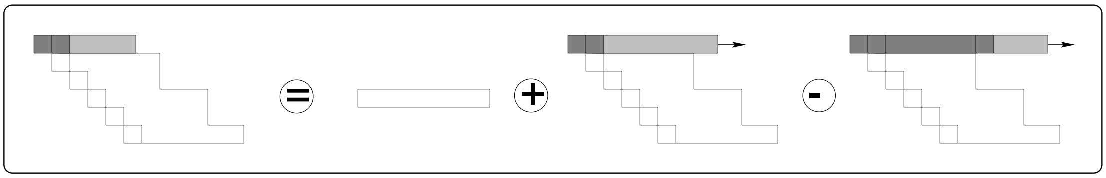

Figure 5: Recursive construction of the LFD polyominoes, by adding rows.

Moreover, the Dyck polyominoes being the LFD polyominoes with width one bases, their height and area generating series $D(t,q)$ is given by

$$D(y,q) = L_1(1,y,q)\,. \tag{15}$$

A straightforward application of Lemma 2.3 from [3] (M. Bousquet-Mélou) gives the solution to the functional equation (13). As we do not need the variable u for our purpose, we set it equal to 1, which simplifies the expression for the generating function.

Proposition 6 *The area and height generating function $L_n(1,y,q)$ for LFD polyominoes having a basis of width n is given by*

$$L_n(1,y,q) = \frac{\displaystyle\sum_{m\geq 0} \frac{(-1)^m y^{m+1} q^{(m+n)(m+1)}}{(q)_m}}{\displaystyle\sum_{m\geq 0} \frac{(-1)^m y^m q^{m(m+1)}}{(q)_m}}\,. \tag{16}$$

■

For $n=1$, this gives

$$D(y,q) = \frac{\displaystyle\sum_{m\geq 0} \frac{(-1)^m y^{m+1} q^{(m+1)^2}}{(q)_m}}{\displaystyle\sum_{m\geq 0} \frac{(-1)^m y^m q^{m(m+1)}}{(q)_m}} \tag{17}$$

for the height and area generating function for Dyck polyominoes. However, we can also express $D(y,q)$ using the classical q-analogue of Catalan numbers $c_n(q)$, satisfying the recurrence

$$c_n(q) = \sum_{k=0}^{n-1} q^k c_k(q) c_{n-1-k}(q)\,, \tag{18}$$

as it is well known that they area-enumerate Dyck paths of length $2n$. The area enumerated by $c_n(q)$ is the number of cells under the path and strictly above its supporting diagonal (i.e. the cells on the diagonal are not included in the area). To get a Dyck polyomino from a Dyck path, we have to add the area of the diagonal. If the length of the path is $2n$, then a factor of q^n has to be added. A further diagonal of cells has to be added because the Dyck paths can touch the supporting diagonal, in which case they are not polyominoes. For the paths of length $2n$, $n+1$ cells thus have to be added, contributing a further q^{n+1} factor to the area. This last diagonal also adds one unit of height to the polyominoes. Hence

$$D(y,q) = \sum_{n\geq 1} y^n q^{2n-1} c_{n-1}(q)\,. \tag{19}$$

3.2.1 Reflective symmetry along the first diagonal

There is a nice area-preserving bijection between d_1-symmetric parallelograms of a given half-perimeter and r^2-symmetric parallelograms with same half-perimeter. Since the minimal rectangle of a d_1-symmetric parallelogram is necessarily a square, the perimeter is a multiple of 4, and thus the half-perimeter is even. Hence we have

$$D_1(x, y, q) = \sum_{k \geq 0} r_{2k}(q) t^{2k} , \tag{20}$$

where $D_1(x, y, q)$ is the generating series of d_1-symmetric parallelogram polyominoes and the $r_{2k}(q)$ are defined by (12). The bijection is shown on an example in Figure 6, and goes as follows: a r^2-symmetric parallelogram has a center of rotation. If it has even half-perimeter, this center will either fall in the center of a cell (if both the height and the width are odd) or be the common corner of four cells forming a square (if both the height and the width are even). In both cases, we consider the first diagonal (parallel to the bissector of the second quadrant) passing through the center of rotation and the region of the parallelogram below the second diagonal (see Figure 6). This region is not a polyomino, but the parallelogram is obtained by glueing the region and a copy of it rotated by 180° in the unique way such that there are no "half-cells" left. Suppose that instead of rotating the copy of the region, we reflect it along the second diagonal and glue it so that there are no half-cells left, then we clearly obtain a d_1-symmetric parallelogram which, further, has the exact same perimeter and area as the initial r^2-symmetric parallelogram. We can similarly reverse the process to start with an arbitrary d_1-symmetric parallelogram and end with a r^2-symmetric parallelogram.

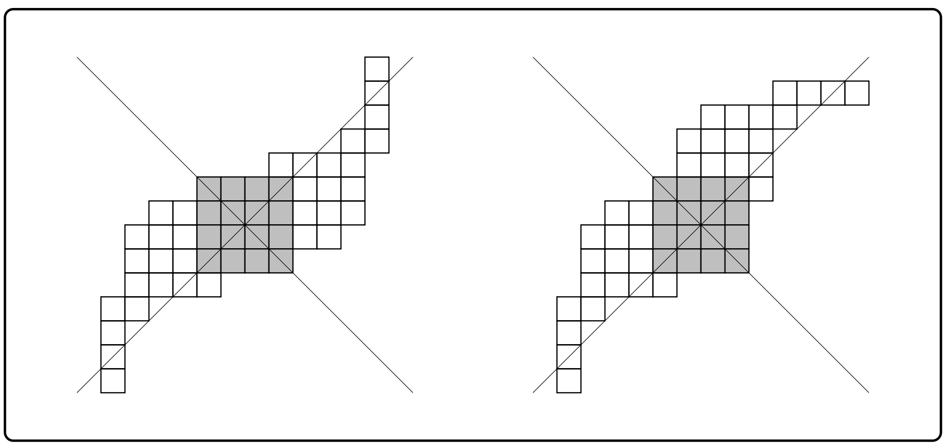

Figure 6: Bijection between r^2-symmetric parallelogram polyominoes of even half-perimeter and d_1-symmetric parallelograms.

3.2.2 Reflective symmetry along the second diagonal

We next consider d_2-symmetric parallelogram polyominoes, i.e. parallelograms which are left invariant by a symmetry along the second diagonal. Figure 7 gives an example of such a parallelogram. We observe first that the minimal rectangle of such a parallelogram will always be a square with side length equal to the quarter-perimeter of the parallelogram.

We note that d_2-symmetric parallelogram polyominoes can be constructed from two copies of a same Dyck polyomino, whose diagonals we glue together (dark cells on Figure 7). The area of the final object will be twice the area of the Dyck polyomino minus the area of diagonal, which was counted twice. There are as many cells on the diagonal as the height of the Dyck polyomino, and the width of the final object will also be the height of the Dyck polyomino. Hence we get

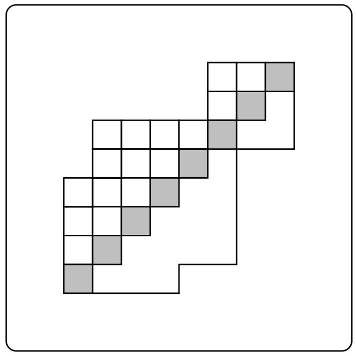

Figure 7: A d_2-symmetric parallelogram polyomino.

Proposition 7 *The generating series $D_2(x, y, q)$ of d_2-symmetric parallelogram polyominoes is given by*

$$D_2(x, y, q) = D\left(\frac{xy}{q}, q^2\right). \tag{21}$$

■

3.2.3 Reflective symmetry along both diagonals

The final (non-cyclic) subgroup whose set of fixed elements we study is the whole group itself. This group is generated by any two nontrivial elements, but it is convenient to consider the symmetries along the two diagonals as the generators. This allows us to characterize the fundamental region of a $\mathfrak{D}_2$-symmetric parallelogram, as can be seen in Figure 8.

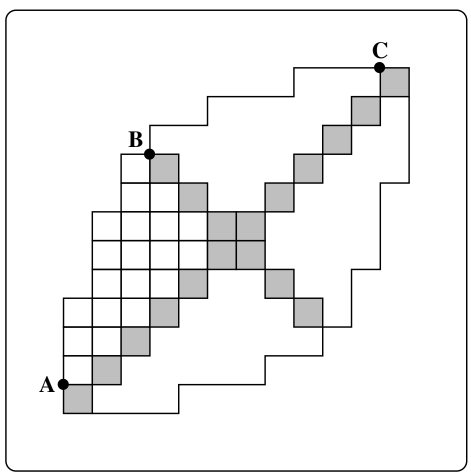

Figure 8: A $\mathfrak{D}_2$-symmetric parallelogram polyomino.

We note first that the minimal rectangle of a $\mathfrak{D}_2$-symmetric parallelogram P is a square. We remark also that the exterior path going from **A** to **C** is a Dyck path that has the additional property that it is symmetric with respect to the second diagonal passing through the center of P (i.e. the Dyck word in x and $\bar{x}$ associated to the Dyck path is a palindrome). Hence P is completely determined by "half" a Dyck path (the path going from **A** to **B**). If P has half-perimeter $2k$ (its half-perimeter is necessarily even since the minimal rectangle is a square), then the path $\mathbf{A} \to \mathbf{C}$ is a symmetrical Dyck path of length $2k - 2$, and the path $\mathbf{A} \to \mathbf{B}$ is simply a left factor of length $k - 1$ of a Dyck path. Thus we have the following result:

Proposition 8 *The number of $\mathfrak{D}_2$-symmetric parallelogram polyominoes of half-perimeter $2k+2$ is given by*

$$d^{(1,2)}_{2k+2}(1) = \begin{cases} \binom{k}{k/2} & \text{if } k \text{ is even}, \\ \frac{1}{2}\binom{k+1}{(k+1)/2} & \text{if } k \text{ is odd}. \end{cases} \tag{22}$$

Proof See Corollary 3. ■

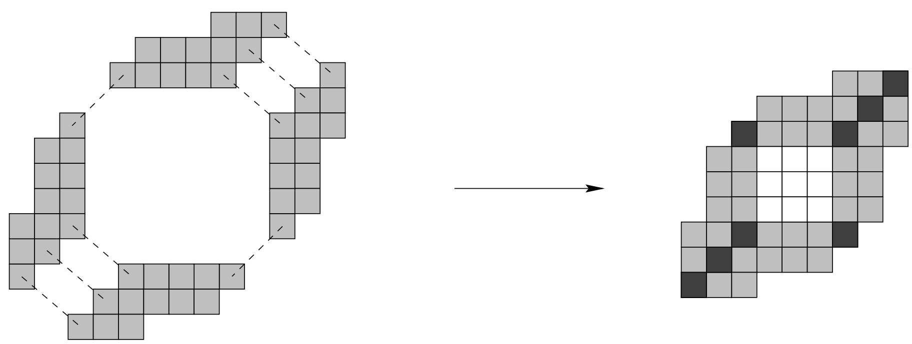

Figure 9: Construction of a $\mathfrak{D}_2$-symmetric parallelogram polyomino from 4 copies of a LFD polyomino.

We obtain the area and half-perimeter generating function for the $\mathfrak{D}_2$-symmetric parallelogram polyominoes by constructing these from 4 copies of a LFD polyomino, as illustrated in Figure 9. Some cells are superposed (the dark ones) and others have to be added (square of white cells in the center), and that has to be taken into account when computing the area of the final object. If the LFD polyomino has area A, height (number of cells on the diagonal) d and base n, then we have that the area of the $\mathfrak{D}_2$-symmetric parallelogram polyomino is $4A - 2d + (n-2)^2 - 2$, while its half-perimeter will be given by $2n + 4(d-1)$. Hence we have the following proposition:

Proposition 9 *The half-perimeter and area generating function $D_{1,2}(t,q)$ for $\mathfrak{D}_2$-symmetric parallelogram polyominoes is given by*

$$D_{1,2}(t,q) = t^2 q + \sum_{n\geq 2} t^{2n-4} q^{n^2-4n+2} L_n(1, \frac{t^4}{q^2}, q^4) \tag{23}$$

where $L_n(u,x,q)$ is the generating series for LFD polyominoes with a base of width n. ■

Corollary 10

$$D_{1,2}(t,q) = t^2 q + \frac{\displaystyle\sum_{n\geq 2}\sum_{m\geq 0} \frac{(-1)^m t^{4m+2n} q^{4m^2+2m+4mn+n^2}}{(q^4)_m}}{1 - t^4 q^6 \displaystyle\sum_{m\geq 0} \frac{(-1)^m t^{4m} q^{4m^2+10m}}{(q^4)_{m+1}}}. \tag{24}$$

Proof This follows from equation (16). ■

Here are the first few terms of $D_{1,2}(t,q)$:

$$D_{1,2}(t,q) = t^2 q + t^4 q^4 + t^6 q^9 + t^8 q^{10} + t^8 q^{14} + t^8 q^{16} + t^{10} q^{15} + t^{10} q^{19} + t^{10} q^{23} + t^{10} q^{25} + \ldots$$

3.3 Congruence-type parallelogram polyominoes

We are now in a position to enumerate congruence-type parallelogram polyominoes, i.e. parallelograms up to rotation and reflection using formula (1) with $G = \mathfrak{D}_2$ and $\mathcal{P} = \mathbb{P}$, the class of all parallelogram polyominoes:

$$|\mathbb{P}/\mathfrak{D}_2|_w = \frac{1}{4} \sum_{g\in\mathfrak{D}_2} |\mathrm{Fix}(g)|_w \,, \tag{25}$$

where $|\mathrm{Fix}(g)|_w$ is the half-perimeter and area generating series of the convex g-symmetric polyominoes. Therefore,

Proposition 11 *The half-perimeter and area generating series $(\mathbb{P}/\mathfrak{D}_2)(t,q)$ of congruence-type parallelograms is given by*

$$(\mathbb{P}/\mathfrak{D}_2)(t,q) = |\mathbb{P}/\mathfrak{D}_2|_w = \frac{1}{4}\left(\mathbb{P}(1,t,t,q) + R_2(t,t,q) + D_1(t,t,q) + D_2(t,t,q)\right) . \tag{26}$$

■

Here are the first few terms of $(\mathbb{P}/\mathfrak{D}_2)(t,q) = \sum_{k\geq 0} \widetilde{p}_k(q)t^k$:

$$\begin{aligned}
\widetilde{p}_2(q) &= q \\
\widetilde{p}_3(q) &= q^2 \\
\widetilde{p}_4(q) &= q^4 + 2\,q^3 \\
\widetilde{p}_5(q) &= q^6 + q^5 + 3\,q^4 \\
\widetilde{p}_6(q) &= q^9 + 2\,q^8 + 3\,q^7 + 4\,q^6 + 6\,q^5
\end{aligned}$$

3.4 Asymmetric parallelogram polyominoes

We can also enumerate asymmetric parallelogram polyominoes, i.e. parallelograms having no symmetry at all, using Möbius inversion in the lattice of subgroups of $\mathfrak{D}_2$.

The reader is referred to [22] for a general discussion of Möbius inversion, and to [18] to see it applied to the enumeration of the symmetry classes of convex polyominoes. We simply give here in Figure 10 the lattice of the subgroups of $\mathfrak{D}_2$ and the value of the Möbius function on the points of the lattice. For subgroups H of $\mathfrak{D}_2$, we denote by $F_{\geq H}$ (resp. $F_{=H}$) the half-perimeter and area generating series for the set of parallelogram polyominoes having at least (resp. exactly) the symmetries of H.

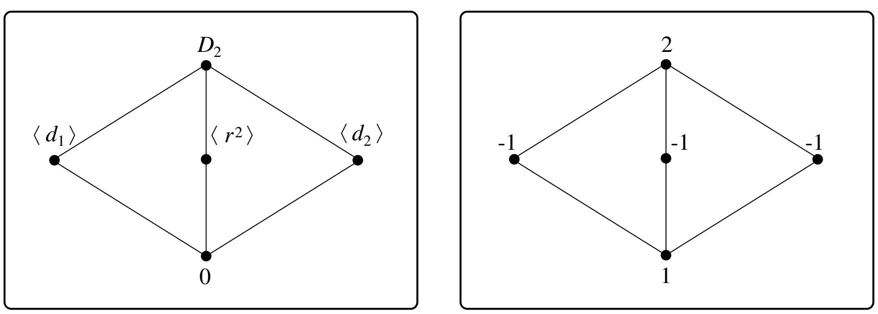

Figure 10: Lattice of the subgroups of $\mathfrak{D}_2$ and the Möbius function $\mu(0,H)$ on the lattice.

Proposition 12 *The half-perimeter and area generating series $\overline{\mathbb{P}}(t,q) = F_{=0}$ of asymmetric parallelogram polyominoes is given by*

$$\begin{aligned}
\overline{\mathbb{P}}(t,q) &= F_{\geq 0} - F_{\geq\langle r^2\rangle} - F_{\geq\langle d_1\rangle} - F_{\geq\langle d_2\rangle} + 2F_{\geq\langle d_1,d_2\rangle} \\
&= \mathbb{P}(1,t,t,q) - R_2(t,t,q) - D_1(t,t,q) - D_2(t,t,q) + 2D_{1,2}(t,t,q)\,. \qquad (27)
\end{aligned}$$

where $D_{1,2}(x,y,q)$ is the generating series of $\mathfrak{D}_2$-symmetric polyominoes. ∎

Here are the first few terms of $\overline{\mathbb{P}}(t,q) = \sum_{k\geq 0} \overline{p}_k(q)t^k$:

$$\begin{aligned}
\overline{p}_2(q) = \overline{p}_3(q) = \overline{p}_4(q) &= 0 \\
\overline{p}_5(q) &= 4\,q^5 + 4\,q^4 \\
\overline{p}_6(q) &= 8\,q^7 + 8\,q^6 + 8\,q^5 \\
\overline{p}_7(q) &= 4\,q^{11} + 8\,q^{10} + 20\,q^9 + 24\,q^8 + 32\,q^7 + 24\,q^6
\end{aligned}$$

Note that the same method would allow us to enumerate the parallelogram polyominoes having exactly the symmetries of any given subgroup of $\mathfrak{D}_2$.

3.5 Asymptotic results

Here we show the asymptotic result that for large area or large perimeter, almost all parallelogram polyominoes are asymmetric. In other words, the probability for a parallelogram polyomino to have at least one symmetry goes to zero as the area or the perimeter goes to infinity.

We know from the work of Pólya ([20]) that the number of parallelogram polyominoes of half-perimeter n is given by

$$p_n^{(t)} = c_{n-1} \sim \frac{4^n}{\sqrt{\pi} n^{3/2}}\,. \tag{28}$$

For the area, we have the result from Bender:

Proposition 13 (**Bender** [1]) *Let $p_n^{(q)}$ be the number of parallelogram polyominoes with area n. Then*

$$p_n^{(q)} \sim k\,\mu^n\,, \tag{29}$$

with

$$k = 0.29745\ldots \qquad \mu = 2.30913859330\ldots$$

∎

Proposition 14 *Let H be any non-trivial subgroup of $\mathfrak{D}_2$ and denote by $P_H^{(q)}(n)$ (resp. $P_H^{(t)}(n)$) the number of H-symmetric parallelogram polyominoes with area (resp. half-perimeter) n. Then,*

$$\lim_{n\to\infty} \frac{P_H^{(q)}(n)}{p_n^{(q)}} = 0 \qquad \text{and} \qquad \lim_{n\to\infty} \frac{P_H^{(t)}(n)}{p_n^{(t)}} = 0\,. \tag{30}$$

Proof The proof for the perimeter part of the Proposition is immediate as we have closed forms for all the coefficients, and thus the limit can be verified to be zero explicitly.

For the area, we need only consider r^2- and d_2-symmetric parallelogram polyominoes, as the d_1-symmetric parallelograms are in bijection with a subclass of r^2-symmetric ones, and the $\mathfrak{D}_2$-symmetric parallelograms are a subclass of all the other classes of symmetry. A same basic argument works for r^2- and d_2-symmetric parallelogram polyominoes, using the fact that they are constructed from two congruent subpolyominoes. A supplementary column sometimes has to be added in the r^2 case, according to whether the width of the initial parallelogram is odd or even. These subpolyominoes are parallelograms in every case, and they have at most half the area of the initial object.

- **r^2-symmetric parallelogram polyominoes:** An r^2-symmetric parallelogram polyomino with even width and area n is constructed from two congruent subparallelograms with exactly half the area, which can be glued together in at most $n/2$ ways (the maximal height of the columns that get glued together). So $P_{r^2}^{(q),even}(n) \leq \frac{1}{2}np_{n/2}^{(q)}$. Hence

$$\lim_{n\to\infty} \frac{P_{r^2}^{(q),even}(n)}{p_n^{(q)}} \quad\leq\quad \lim_{n\to\infty} \frac{\frac{1}{2}nk\mu^{n/2}}{k\mu^n} \quad=\quad 0\,.$$

Next consider an r^2-symmetric parallelogram polyomino with odd width and area n. This polyomino is constructed from a central column (n choices of height) and two congruent subparallelograms of area at most $\lfloor n/2 \rfloor$. Then there are at most n possible positions where to glue the subparallelograms to the central column (they are glued symmetrically). Thus $P_{r^2}^{(q),odd}(n) \leq n^2(1+p_1^{(q)}+p_2^{(q)}+\ldots+p_{\lfloor n/2 \rfloor}^{(q)}) < n^3 p_{\lfloor n/2 \rfloor}^{(q)}$ and the result follows as above. Hence the result holds for the subgroup $\langle r^2 \rangle$ of $\mathfrak{D}_2$;

- **d_2-symmetric convex polyominoes:** Let P be a d_2-symmetric parallelogram polyomino and Q its fundamental region. Suppose that P has b cells on the diagonal symmetry axis. Then the minimum area P can have is $b+2(b-1)$. This gives a minimum area of $b+(b-1)$ for Q. Hence

$$\frac{\text{Area of } Q_{min}}{\text{Area of } P_{min}} = \frac{2b-1}{3b-2}.$$

Then if we add to Q a cell not on the diagonal symmetry axis, two cells get added to P, and thus we conclude that the ratio can only decrease as we make P into a larger d_2-symmetric parallelogram polyomino with the same number of cells on the diagonal axis. For $b \geq 2$, the ratio will be smaller than or equal to $3/4$. As a loose approximation, we can take Q to be any parallelogram polyomino of area at most $\lceil 3n/4 \rceil$. Also, for $n > 1$, P will necessarily have more than one cell on its diagonal. This gives $P_{d_2}(n) \leq 1 + p_1^{(q)} + p_2^{(q)} + \ldots + p_{\lceil 3n/4 \rceil}^{(q)} \leq n\, p_{\lceil 3n/4 \rceil}^{(q)}$. Hence the result will also hold for the subgroup $\langle d_2 \rangle$.

■

Proposition 15 *If we denote by $\overline{p}_n^{(q)}$ (resp. $\overline{p}_n^{(t)}$) the number of asymmetric parallelogram polyominoes of area (resp. half-perimeter) n, then*

$$\overline{p}_n^{(q)} \sim p_n^{(q)}, \qquad (31)$$

$$\overline{p}_n^{(t)} \sim p_n^{(t)}. \qquad (32)$$

Proof We get the result from equation (27) and from the previous Proposition. ■

Two tables can be found in the appendix that present the numbers of parallelogram polyominoes according to their symmetry types and their perimeter or area. The columns indexed by subgroups of $\mathfrak{D}_2$ give the numbers of parallelogram polyominoes of a given perimeter or area that are left fixed by the symmetries of the subgroup. The columns *# Orbits* and *Asym* give respectively the number of congruence-type and asymmetric parallelogram polyominoes of the given size.

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Appendix

Half-perimeter	$\lvert\mathrm{Fix}(1)\rvert$	$\lvert\mathrm{Fix}(r^2)\rvert$	$\lvert\mathrm{Fix}(d_1)\rvert$	$\lvert\mathrm{Fix}(d_2)\rvert$	# Orbits	$\lvert\mathrm{Fix}(\mathfrak{D}_2)\rvert$	Asym
2	1	1	1	1	1	1	0
3	2	2	0	0	1	0	0
4	5	3	1	3	3	1	0
5	14	6	0	0	5	0	8
6	42	10	2	10	16	2	24
7	132	20	0	0	38	0	112
8	429	35	5	35	126	3	360
9	1430	70	0	0	375	0	1360
10	4862	126	14	126	1282	6	4608
11	16796	252	0	0	4262	0	16544
12	58786	462	42	462	14938	10	57840
13	208012	924	0	0	52234	0	207088
14	742900	1716	132	1716	186616	20	739376
15	2674440	3432	0	0	669468	0	2671008
16	9694845	6435	429	6435	2427036	35	9681616
17	35357670	12870	0	0	8842635	0	35344800
18	129644790	24310	1430	24310	32423710	70	129594880
19	477638700	48620	0	0	119421830	0	477590080
20	1767263190	92378	4862	92378	441863202	126	1767073824

Table 1: Parallelogram polyominoes enumerated by their symmetries and half-perimeter.

Area	$\lvert\mathrm{Fix}(1)\rvert$	$\lvert\mathrm{Fix}(r^2)\rvert$	$\lvert\mathrm{Fix}(d_1)\rvert$	$\lvert\mathrm{Fix}(d_2)\rvert$	# Orbits	$\lvert\mathrm{Fix}(\mathfrak{D}_2)\rvert$	Asym
1	1	1	1	1	1	1	0
2	2	2	0	0	1	0	0
3	4	2	0	2	2	0	0
4	9	5	1	1	4	1	4
5	20	4	0	4	7	0	12
6	46	12	0	2	15	0	32
7	105	9	1	9	31	1	88
8	242	28	0	6	69	0	208
9	557	21	1	21	150	1	516
10	1285	65	1	13	341	1	1208
11	2964	48	0	48	765	0	2868
12	6842	152	2	32	1757	0	6656
13	15793	111	1	111	4004	1	15572
14	36463	351	1	73	9222	1	36040
15	84187	257	3	257	21176	1	83672
16	194388	814	2	172	48844	2	193404
17	448847	593	3	593	112509	1	447660
18	1036426	1882	4	396	259677	0	1034144
19	2393208	1370	4	1370	598988	2	2390468
20	5526198	4352	6	920	1382869	2	5520924
21	12760671	3165	7	3165	3191752	1	12754336
22	29466050	10054	8	2124	7369559	2	29453868
23	68041019	7309	11	7309	17013912	3	68026396

Table 2: Parallelogram polyominoes enumerated by their symmetries and area.

Résumé substantiel

Les polyominos parallélogrammes sont la sous-classe des polyominos convexes (horizontalement et verticalement) dans le réseau carré consistuée de ceux qui touchent les coins inférieur-gauche et supérieur-droit de leur rectangle minimal (le plus petit rectangle les contenant). Voir la figure 1.

Il existe une littérature importante sur l'énumération de nombreuses classes de polyominos possédant diverses propriétés de convexité et de directionalité, car ils apparaissent en combinatoire, en physique statistique, en informatique théorique et aussi en mathématiques récréatives (voir M. Bousquet-Mélou [5] pour une récente synthèse). L'étude des polyominos parallélogrammes selon le demi-périmètre et l'aire a débuté avec Pólya, puis d'autres ont réussi à donner des énumérations plus fines, en particulier Bender, Klarner, Rivest, Delest, Fédou, Viennot, etc. Mireille Bousquet-Mélou, en utilisant la méthodologie de Temperley ([25]), a pu donner une série génératrice selon la hauteur, la largeur, l'aire et les hauteurs des première et dernière colonnes des polyominos parallélogrammes ([3], [5]).

Comme pour les polyominos généraux, l'aire d'un polyomino parallélogramme est le nombre de cellules (cases) qu'il contient, et son demi-périmètre est la somme de sa largeur et de sa hauteur. Deux polyominos sont généralement considérés équivalents si l'un peut être obtenu de l'autre par une translation dans le réseau (voir D. A. Klarner [15]). Il est aussi naturel de vouloir considérer les polyominos *à congruences-près*, c'est-à-dire les classes d'équivalence de polyominos sous rotations et réflexions, comme si l'on considérait les polyominos comme des pièces dans l'espace (voir S. W. Golomb [14]). Les polyominos convexes ont été énumérés à congruences-près dans [18], selon l'aire et le demi-périmètre, et dans [21], Redelmeier a énuméré les polyominos généraux à congruences-près, selon l'aire, jusqu'à aire 24, en utilisant l'ordinateur.

En faisant usage de la formule de Cauchy-Frobenius (ou lemme de Burnside), le problème se réduit à l'énumération des orbites de polyominos parallélogrammes sous l'action d'un sous-groupe du groupe diédral $\mathfrak{D}_4$, le groupe des symétries du carré. Le sous-groupe en question, noté $\mathfrak{D}_2$, est le sous-groupe des symétries du carré qui laissent la classe $\mathbb{P}$ des polyominos parallélogrammes stable sous leur action. Ce sous-groupe est donc $\mathfrak{D}_2 = \langle r^2, d_1 \rangle = \{1, r^2, d_1, d_2\}$, où r^2 est une rotation de 180°, et d_1 et d_2 les réflexions par rapport aux deux diagonales du carré (par convention d_2 est la réflexion par rapport à la bissectrice du premier quadrant). La formule de Cauchy-Frobenius permet de relier le nombre d'orbites de l'action d'un groupe fini sur un ensemble fini aux nombres d'éléments laissés fixes par chacune des symétries du groupe (formule (1)). La formule se généralise aux cardinalités pondérées, si la fonction de poids est invariante sous l'action du groupe, ce qui permet de remplacer les nombres de polyominos dans la formule par les séries génératrices selon l'aire et le demi-périmètre des classes correspondantes, puisque l'aire et le demi-périmètre sont invariants sous les rotations et les réflexions.

Cet article présente d'abord l'énumération selon l'aire et le demi-périmètre de chacune des classes de symétries de l'action de $\mathfrak{D}_2$ sur $\mathbb{P}$, ce qui permet ensuite par la procédure décrite ci-haut de compter les polyominos parallélogrammes à congruences-près (les orbites de l'action). Finalement, en utilisant l'inversion de Möbius dans le treillis des sous-groupes de $\mathfrak{D}_2$, les polyominos parallélogrammes asymétriques, c'est-à-dire ceux qui ne sont laissés fixes que par l'élément identité du groupe, sont énumérés, et il est ensuite prouvé que, peu importe s'ils sont comptés selon leur aire ou leur demi-périmètre, ils sont asymptotiquement aussi nombreux que les polyominos parallélogrammes (ce qui confirme l'intuition).

On remarque que l'énumération de la plupart des classes de symétries selon le demi-périmètre fait intervenir sous une forme ou une autre des chemins de Dyck (ou mots de Dyck; voir J. Labelle [17]), dénombrés par les nombres de Catalan c_n, ou des facteurs-gauche de chemins de Dyck, dénombrés par les coefficients binomiaux centraux b_n (voir Cori et Viennot [8]). Lorsque l'aire est prise en compte, des q-analogues de ces nombres apparaissent naturellement, certains connus, d'autres nouveaux.

La section 1 de l'article est une introduction à l'étude des polyominos parallélogrammes, dont l'essentiel a été repris dans ce résumé.

La section 2 présente une preuve classique du fait que les polyominos parallélogrammes sont énumérés selon le demi-périmètre par les nombres de Catalan, qui consiste en une bijection entre les polyominos parallélogrammes et les chemins de Dyck. On y cite ensuite le résultat de M. Bousquet-Mélou donnant la série génératrice des polyominos parallélogramme selon leur hauteur, leur largeur et leur aire.

La section 3 contient les résultats. Les sous-sections 3.1 et 3.2 correspondent aux symétries par rotations (r^2) et par réflexions (d_1 et d_2) respectivement. Cette dernière sous-section débute par l'introduction d'une nouvelle structure, les polyominos LFD (polyominos correspondant à des facteurs-gauche de chemins de Dyck), puis l'énumération des polyominos parallélogrammes laissés fixes par d_1 et d_2 est accomplie en 3.2.1 et 3.2.2 respectivement. Une classe de symétries supplémentaire est dénombrée en 3.2.3 : celle des polyominos parallélogrammes laissés fixes à la fois par les deux réflexions diagonales. Cette classe n'est pas nécessaire au dénombrement des polyominos parallélogrammes à congruences-près, mais elle l'est au dénombrement des polyominos parallélogrammes asymétriques.

La sous-section 3.3 donne ensuite le dénombrement des polyominos parallélogrammes à congruences-près, en utilisant le lemme de Burnside. La sous-section 3.4 donne le dénombrement des polyominos parallélogrammes asymétriques, en utilsant l'inversion de Möbius. Finalement, la sous-section 3.5 donne la preuve que les polyominos parallélogrammes asymétriques sont asymptotiquement aussi nombreux que les polyominos parallélogrammes.

Une annexe donne en deux tables les premiers termes des dénombrements des diverses classes de symétries, des polyominos parallélogrammes à congruences-près et asymétriques. La première table donne les premiers termes selon le demi-périmètre, la deuxième selon l'aire.

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